UNIFORMITY IN LINEAR SPACES*

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Introduction

The first chapter of this paper concerns itself with questions of uniform boundedness of sets of points in a Banach space and sets of functionals on a Banach space, as well as with a group of closely related resonance theorems. A well known example coming under this heading is the theorem of Toeplitz [36]† stating that $\sup_m \sum_{n=1}^\infty |a_{mn}| < \infty$ providing $\eta_m = \sum_{n=1}^\infty a_{mn}\xi_n$ converges whenever ξ_n does. Another is the theorem of Hahn [12] stating that if an arbitrary continuous function has the partial sums of its Fourier expansion, with respect to an orthonormal sequence of bounded functions ω_n, essentially bounded, then the sequence $\int |\sum_{\nu=1}^n \omega_\nu(x)\omega_\nu(t)|\, dt$ is also essentially bounded. Still another is the theorem stating that if the adjoint of an everywhere defined transformation between Banach spaces is everywhere defined, then the transformation is continuous. This was proved, at least for Hilbert space, by von Neumann [21], Stone [34], Tamarkin ([34], p. iv), and Stone and Tamarkin [35] and is probably not usually thought of as a theorem on uniform boundedness.

Most of the results we have in the first chapter are new, others have been proved only in special cases, and some are well known but the proofs heretofore given have been different. Previous methods for discussing questions of uniform boundedness divide themselves into three groups (i) those associated with the names of Lebesgue [18], Banach [2], Hahn [11], and Hildebrandt [13]; (ii) those characterized by the elegant and direct use of the Baire category theorem as in the works of Banach [1], Saks [31], Saks and Tamarkin [32], and others; and (iii) those employed in a recent theorem of Gelfand [9] the proof of which is closely related to that of the category theorem.

The second chapter of this paper is concerned with cases where a weak limiting process implies a strong one. These cases seem to be rather rare but probably more will come to light in the future. The theorem quoted above, stating that the existence of the adjoint implies the continuity of the function, belongs to the class of questions discussed in Chapter II as well as those discussed in Chapter I, while its analogue (Theorem 42) in the Boolean ring of

* Presented to the Society, December 31, 1936, under the title, *Integration of vector-valued functions*; received by the editors September 2, 1937.

† Numbers in brackets refer to the bibliography at the end of the paper.

Lebesgue measurable subsets of (0, 1), which states that an additive vector valued set function $y(e)$ is absolutely continuous providing $\gamma y(e)$ is absolutely continuous for every linear functional γ, is more characteristic of the phenomenon discussed in the second chapter. This is because the continuity of an additive function on a Boolean ring is not a consequence of its boundedness as is the case with linear operations on a Banach space. The theorem on the ring is closely related to the theorem of Orlicz [23] which asserts that for weakly complete Banach spaces the weak unconditional convergence of a series implies its unconditional convergence in the strong sense. Another result in this category which has appeared recently is the theorem of Pitt [26] which states that a bilinear form $\sum \xi_i a_{ij} \eta_j$, if bounded for

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |\xi_i|^p \right)^{1/p} = \|y\|_q = \left(\sum_{j=1}^{\infty} |\eta_j|^q \right)^{1/q} = 1,$$

where $1/p+1/q<1$, is convergent in the sense of Pringsheim as a double series and uniformly for $\|x\|_p=\|y\|_q=1$. This result is not true for $1/p+1/q \geqq 1$. The theorem may be worded in terms of linear operations on l_q and would state that every continuous linear operator on l_q to $l_{p'}$ is completely continuous if $q>p' \geqq 1$. In this form it is closely related to Theorem 71 which has also been proved by Pettis [24] and states that an operator on L^p to L, $(p>1)$, which has the form

$$\psi(s) = \int_0^1 H(s, t)\phi(t)dt,$$

where $H(s, t)$ is in $L^{p'}$ for each s, is necessarily completely continuous. This shows that the expansion of ψ with respect to any complete orthonormal sequence in L is necessarily convergent (in L) uniformly with respect to $\int_0^1 |\phi(t)|^p dt = 1$.

With the exception of Pitt's result in the case where $p'>1$, Chapter II contains results considerably broader than those outlined above. Another example of the above type, which falls, however, more naturally into the first chapter is the result that the Riemann integral $\int_0^1 \phi(t)df(t)$, where $f(t)$ has its values in a Banach space, exists for every continuous function ϕ providing the integral $\int_0^1 \phi(t)d\gamma f(t)$ exists for every continuous ϕ and every linear functional γ. Still another is that (Theorem 76) a function $f(z)$ from a domain in the complex z-plane to a complex Banach space is analytic providing $\gamma f(z)$ is analytic for every complex valued linear functional γ.

In Chapter III the results of the preceding chapters are applied to a theory of Lebesgue integration which is broader than those heretofore given and which furnishes the natural tool for the solution of a general type of moment problem (Theorem 61). The central idea underlying the integral is that a function $f(p)$ (on a measurable set E to a Banach space Y) which has the property that $\gamma f(p)$ is summable for every γ in a closed linear manifold $\Gamma \infty \overline{Y}$, defines uniquely a point in $\overline{\Gamma}$ according to the equation

$$\bar{\gamma}_E(\gamma) = \int_E \gamma f(p) dp.$$

This notion is intimately related to the recent work of Pettis whose manuscript I had the privilege of seeing shortly before I finished the typing of my own. Pettis' paper gives the reader a very interesting discussion of the case where $\Gamma = \overline{Y}$ and $\bar{\gamma}_e$ is in Y for every measurable subset e of E, and this case is undoubtedly one of the most important to be considered. Another case of interest which we have discussed only slightly is the case where Y itself is a conjugate space $\overline{Z}$, and $\Gamma = Z$. Here one always has $\bar{\gamma}_e$ in Y. The integral in this case has been defined by Gelfand [9] but has not, as far as we know, been discussed in any detail.

In Chapter IV a few instances of the general theorems are pointed out.

Notation. The notation we have used is for the most part self explanatory or else explained where it is introduced, but it might help the reader to keep in mind that throughout the paper Y and Z are arbitrary Banach spaces while X is a Banach space subjected to three restrictions imposed at the beginning of Chapter I and later in Chapter II to a fourth restriction. The symbol Γ is always used for a closed linear manifold in $\overline{Y}$, and γ for a point in Γ. Thus in expressions such as $\sup_{\|\gamma\|=1} \gamma y$ it is to be understood, unless explicitly stated to the contrary, that γ is restricted to be in Γ.

We will be dealing with functions $f(t)$ on a class T to a Banach space Y. In connection with these functions the following symbols occur:

$$f(t), \qquad f, \qquad \gamma f(.), \qquad \nu f.$$

The first quite naturally means the value of the function for the argument t. The second is used when we think of f as a point in an abstract space. The third, following the notation of E. H. Moore, we are to interpret as follows: It may be that for a given γ in Γ the numerical function $\gamma f(t)$ on T, when considered as a single entity, is an element of a Banach space X whose elements are numerical functions on T. If so, then $\gamma f(.)$ represents the point in the Banach space X. In the last one ν is a linear functional on a Banach space X whose elements are numerical functions on T. In Chapter III we have shown that the domain of a linear functional ν on X can be extended in a natural way to the class $\mathfrak{X}[Y, \Gamma]$ of all $f(t)$ on T to Y such that $\gamma f(.)$ is in X for every γ

in Γ. The symbol νf is then the value of the function ν for the argument f in $\mathfrak{X}[Y, \Gamma]$. This notation might be confusing in case $\mathfrak{X}[Y, \Gamma]$ were a Banach space (which it sometimes is), and we wished to express the value of a linear functional on $\mathfrak{X}[Y, \Gamma]$ for a particular argument in $\mathfrak{X}[Y, \Gamma]$. But since we shall not have occasion to do this, no confusion should arise. It should be noted that if $X=Y$, then both symbols $\gamma f(.)$ and γf may have a meaning, but the meaning is in general different.

Chapter I

1.0. **Uniform boundedness.** Let X be a Banach space composed of numerical functions $\phi(t)$, where t ranges over an abstract set T. Throughout what follows X will be subject to the following conditions:

(1) *If* $\phi_1(t)+\phi_2(t)=\phi(t)$ *for* t *in* T, *then* $\phi_1+\phi_2=\phi$.

(2) *If, for a numerical constant* c, $c\phi_1(t)=\phi(t)$ *for* t *in* T, *then* $c\phi_1=\phi$.

(3) *If* $\phi_n\to\phi$ *and* $\phi_n(t)\to\phi_*(t)$ *for* t *in* T, *then* $\phi=\phi_*$.

Let Y be an arbitrary Banach space and Γ a closed linear manifold in $\overline{Y}$, the space conjugate to Y. The linear space $\mathfrak{X}=\mathfrak{X}[Y, \Gamma]$ is, by definition, the space of all functions $f=f(t)$ on T to Y such that $\gamma f(.)$ is in X for every γ in Γ.

Theorem 1. *If f is in* $\mathfrak{X}[Y, \Gamma]$, *then* $\gamma f(.)$ *is a continuous linear operation on* Γ *to* X. *In other words, there exists a smallest non-negative number* $\|f\|$ *such that*

$$\|\gamma f(.)\|_X \leqq \|f\|\cdot\|\gamma\|, \qquad \gamma\in\Gamma.$$

By (1) and (2) we see that the operation $U(\gamma)=\gamma f(.)$ on Γ to X is additive, that is,

$$U(c_1\gamma_1 + c_2\gamma_2) = c_1U(\gamma_1) + c_2U(\gamma_2),$$

for every pair c_1, c_2 of numerical constants and every pair γ_1, γ_2 of points in Γ. Condition (3) shows that if $\gamma_n\to\gamma^*$ and $U(\gamma_n)\to\phi$, then $U(\gamma^*)=\phi$; thus by a well known theorem ([1], p. 41, Theorem 7), U is a continuous operation from Γ to X.

Theorem 2. *Let $f(t)$ be a function on an arbitrary set T to a Banach space Y. If*

$$\sup_{t\in T} |\gamma f(t)| < \infty, \qquad \gamma\in\overline{Y},$$

then

$$\sup_{t\in T} \|f(t)\| < \infty.$$

Since $\sup \gamma y=\|y\|$ ([1], p. 55, Theorem 3) where the sup is taken over all γ in $\overline{Y}$ for which $\|\gamma\|=1$, this theorem follows from Theorem 1 by taking $\Gamma=\overline{Y}$ and $X=M^*(T)$ = the space of functions bounded on T.

THEOREM 3. *Let Y be a Banach space and f_t a function on an arbitrary range T to $\overline{Y}$, the space conjugate to Y. If*

$$\sup_{t \in T} |f_t y| < \infty, \qquad y \in Y,$$

then

$$\sup_{t \in T} \|f_t\| < \infty .$$

This follows from Theorem 1 by taking $X = M^*(T)$ and replacing Y by $\overline{Y}$ and Γ by Y which can be considered as a closed linear manifold in $\overline{\overline{Y}}$ and defined as all $\bar{\bar{y}}$ in $\overline{\overline{Y}}$ expressible in the form $\bar{\bar{y}}(\bar{y}) = \bar{y}(y)$.

Occasionally in what follows we shall assume that there is a notion of null set in T. This notion is subject to the single restriction:

(N) *A denumerable sum of null sets is a null set.*

Thus the notion of null set may be defined in terms of a completely additive measure function, in terms of first category sets, or in terms of sets consisting of at most a denumerable number of points, and so on, or may simply mean a void set. Such terms as ess. $\sup._t \phi(t)$ and the Banach space $M(T)$ of essentially bounded functions on T then have a meaning.

THEOREM 4. *Let T be a set (t) of points in which there is a notion of null set satisfying condition* (N). *Let Y be a Banach space for which $\overline{Y}$ is separable. If $f(t)$ on T to Y is such that $\gamma f(t)$ is essentially bounded for every γ in $\overline{Y}$, then $\|f(t)\|$ is essentially bounded.*

In Theorem 1 take $X = M(T)$, $\Gamma = \overline{Y}$; then

$$\text{ess.} \sup_t . \ |\gamma f(t)| \leqq \|f\| \cdot \|\gamma\| .$$

If $\{\gamma_i\}$ is dense in $\overline{Y}$ and T_i is the set in T where

$$|\gamma_i f(t)| > \|f\| \cdot \|\gamma_i\| ,$$

then T_i, and thus $\sum T_i$, is a null set, and

$$|\gamma f(t)| \leqq \|f\| \cdot \|\gamma\|$$

for every γ and every t in $T - \sum T_i$. Thus $\|f(t)\| \leqq \|f\|$ on $T - \sum T_i$.

THEOREM 5. *Let T be a set (t) of points in which there is defined a notion of null set satisfying condition* (N). *Let Y be a separable Banach space. If f_t on T to $\overline{Y}$ is such that $f_t y$ is essentially bounded for every y in Y, then $\|f_t\|$ is essentially bounded.*

The proof is entirely analogous to that of Theorem 4.

THEOREM 6. *Let V be an arbitrary set (v) of points and U a set (u) in which there is a notion of null set satisfying condition* (N). *Let Y be a Banach space for which $\overline{Y}$ is separable. If $f(u, v)$ on UV to Y is such that for each γ in $\overline{Y}$ there is a null set $U_\gamma \subset U$ and a constant M_γ such that*

$$|\gamma f(u, v)| \leqq M_\gamma, \qquad u \in U - U_\gamma, v \in V,$$

then there is a null set $U_0 \subset U$ and a constant M such that

$$\|f(u, v)\| \leqq M, \qquad u \in U - U_0, v \in V.$$

In the product space $T = UV$ of points (u, v), null sets may be defined as sets of the form U_0V, where U_0 is a null set. Theorem 6 is then a corollary of Theorem 4.

THEOREM 7. *Let U and V be as in Theorem 6 and Y a separable Banach space. If $f_{u,v}$ on UV to $\overline{Y}$ is such that for each y in Y there is a null set $U_y \subset U$ and an M_y such that*

$$|f_{u,v}y| \leqq M_y, \qquad u \in U - U_y, v \in V,$$

then there is a null set $U_0 \subset U$ and a constant M such that

$$\|f_{u,v}\| \leqq M, \qquad u \in U - U_0, v \in V.$$

This follows from Theorem 5 as Theorem 6 did from Theorem 4.

The postulate of separability in Theorems 4 and 6 is not entirely necessary. They hold, for example, when $Y = L$, the space of summable functions, and thus $\overline{Y} = M$, the space of essentially bounded and measurable functions, which is not separable. They hold even when $Y = M$ and $\overline{Y}$ is therefore the space of bounded additive set functions ([7], [15]). These facts we shall prove presently.

Indeed they hold (in a more general form in that γ is not arbitrary) for any space Y for which $\overline{Y}$ is the first or second conjugate of a separable space. As the reader will readily see from the argument, the only thing necessary is that $\overline{Y}$ have the following property: Let Y be a Banach space; then the conjugate space $\overline{Y}$ is said to be a *fundamentally separable* space (f.s. space) with determining manifold Γ in case Γ is a separable closed linear manifold in $\overline{Y}$ such that for every y in Y and $\epsilon > 0$ there is a γ in Γ with $\|\gamma\| = 1$ and $\gamma y > \|y\| - \epsilon$; that is, $\sup_{\|\gamma\|=1} \gamma y = \|y\|$ for each y.

THEOREM 8. *If Y is a separable Banach space, then $\overline{Y}$ and $\overline{\overline{Y}}$ are fundamentally separable spaces.*

The space $\overline{Y}$ is an f.s. space, for let y_p be dense in Y, and let γ_p in $\overline{Y}$ be such that ([1], p. 55, Theorem 3)

$$\|\gamma_p\| = 1, \qquad \gamma_p y_p = \|y_p\|.$$

Let Γ be the closed linear manifold determined by γ_p, $(p=1, 2, \cdots)$. Then if $\|y_p - y\| < \epsilon/2$, we have

$$\sup_{\|\gamma\|=1} \gamma y = \sup_{\|\gamma\|=1} [\gamma(y - y_p) + \gamma y_p] > \|y_p\| - \epsilon/2 > \|y\| - \epsilon.$$

In the case of $\overline{\overline{Y}}$ we can take $\Gamma = Y$.

A determining manifold for M, the space of essentially bounded functions, is the subspace C of continuous functions. This is a consequence of the formula

$$\int f'(t)g(t)dt = \int g(t)df$$

(where $f(t)$ is an absolutely continuous function, $g(t)$ is a continuous function, and the integral on the right is the Riemann-Stieltjes integral) together with the fact that

$$\sup_{|g(t)|\leq 1} \int gdf = (\text{total variation of } f) = \int |f'(t)| dt.$$

A determining manifold for the space M of additive set functions $\phi(E)$, with $\|\phi\|$ = total variation of ϕ, is the subspace consisting of the absolutely continuous set functions. This set is isomorphic to L. Similarly, a determining manifold for the space BV of functions f, which are of bounded variation and are normalized so that the total variation of f is the norm of the linear functional $\int gdf$ on C, is the subspace of absolutely continuous functions. As a final example of a non-separable f.s. space we mention the space $\overline{BV}$ [16]. A determining manifold is the space C itself, that is, the set of linear functionals on BV which are expressible in the form

$$\gamma(f) = \int g(t)df,$$

where g is continuous and $\|\gamma\| = \sup_t |g(t)|$.

Theorems 4 and 6 become then, in their more general form, Theorems 9 and 10 below.

THEOREM 9. *Let T be a set (t) of points in which there is a notion of null set satisfying condition* (N). *Let $\overline{Y}$ be an f.s. space with determining manifold Γ. If $f(t)$ on T to Y is such that $\gamma f(t)$ is essentially bounded for every γ in Γ, then $\|f(t)\|$ is essentially bounded.*

THEOREM 10. *Let V be an arbitrary set (v) of points and U a set (u) in which there is a notion of null set satisfying condition* (N). *Let $\overline{Y}$ be an f.s. space with determining manifold Γ. If $f(u, v)$ on UV to Y is such that for each γ in Γ there is a null set $U_\gamma \subset U$ and an M_γ such that*

$$|\gamma f(u, v)| \leqq M_\gamma, \qquad u \in U - U_\gamma,\ v \in V,$$

then there is a null set $U_0 \subset U$ and a constant M such that

$$\|f(u, v)\| \leqq M, \qquad u \in U - U_0,\ v \in V.$$

1.1. **Functions of bounded variation.** Let Δ be a finite number of non-overlapping intervals (a_i, b_i) on (a, b). A function $f(P)$ on (a, b) to the Banach space Y is said to be of *bounded variation** on (a, b) in case

$$\sup_\Delta \|\Delta f\| < \infty,$$

where $\Delta f = \sum (f(b_i) - f(a_i))$. This is equivalent, in the case of numerical functions, to saying that the sum $\sum |f(b_i) - f(a_i)|$ is bounded in Δ. This notion of bounded variation is too broad for some purposes. For instance in non-separable spaces a function may be of bounded variation and nowhere continuous, for example, $f(t)$ on $(0, 1)$ to the space of bounded functions defined as

$$f(t) = K(s, t) = \begin{cases} 1, & 0 \leqq s \leqq t, \\ 0, & t < s \leqq 1. \end{cases}$$

This cannot happen in separable spaces. But regardless of the space we have the following theorem:

THEOREM 11. *Let f be a function of bounded variation on the interval (a, b) to a Banach space Y. Then the Riemann-Stieltjes integral $\int_a \phi(s) df(s)$ exists for every continuous function ϕ.*

Let $\pi^1 = (\delta_n^1)$, $\pi^2 = (\delta_m^2)$ be two partitions of (a, b) with norm so small that the oscillation of ϕ on any δ_n^1 or δ_m^2 is less than ϵ. Then if $\tau_n^1 \in \delta_n^1$ and $\tau_m^2 \in \delta_m^2$,

$$\begin{aligned} \sum_n \phi(\tau_n^1)\delta_n^1 f - \sum_m \phi(\tau_m^2)\delta_m^2 f &= \sum_n \phi(\tau_n^1) \sum_m (\delta_n^1 \delta_m^2 f) - \sum_m \phi(\tau_m^2) \sum_n (\delta_m^2 \delta_n^1 f) \\ &= \sum_n \sum_m [\phi(\tau_n^1) - \phi(\tau_m^2)](\delta_n^1 \delta_m^2 f), \end{aligned}$$

where $\delta_n^1 \delta_m^2$ is the ordinary product of the two intervals δ_n^1, δ_m^2, and $\delta_n^1 \delta_m^2 f = 0$ in case $\delta_n^1 \delta_m^2$ is empty.

Now to show that this sum is in norm less than or equal to $2\epsilon \sup_\pi \|\sum_\pi \delta f\|$, which is all that is needed, the following lemma will suffice:

* See Gelfand [9] where a corresponding notation is introduced.

LEMMA. *Let $a=(a_1, a_2, \cdots, a_n)$ be a variable point in n-space with norm $\|a\| = \sup_i |a_i|$. Let $y_1, \cdots, y_n$ be n fixed points in a Banach space Y. Then the linear operation*

$$U(a) = \sum_{i=1}^{n} a_i y_i$$

on n-space to Y has its norm

$$\|U\| \leqq 2 \sup_{\sigma} \left\| \sum_{i \in \sigma} y_i \right\|,$$

where σ stands for any set of integers between one and n.

By a well known theorem ([1], p. 55, Theorem 3)

$$\|U\| = \sup_{\|\gamma\|=1} \|\gamma U\|, \quad \text{where} \quad \gamma \in \overline{Y}.$$

Now $\gamma U a = \sum_{i=1}^{n} \gamma a_i y_i$, and

$$\|\gamma U\| = \sum_{i=1}^{n} |\gamma y_i| = \sum_{\sigma_\gamma^+} \gamma y_i - \sum_{\sigma_\gamma^-} \gamma y_i$$

$$\leqq 2\|\gamma\| \sup_{\sigma} \left\| \sum_{i \in \sigma} y_i \right\|,$$

where σ_γ^+ (σ_γ^-) is the set of integers for which $\gamma y_i \geqq 0$ (<0). This completes the proof of the existence of $\int \phi df$.

Consider the space BV_0 of numerical functions $\phi(P)$ of bounded variation on (a, b) with $\phi(a)=0$* and with $\|\phi\| = \int_a |d\phi|$.

THEOREM 12. *If $f(P)$ on (a, b) to the Banach space Y is such that $\gamma f(.)$ is in BV_0 for every γ in $\overline{Y}$, then f is of bounded variation on (a, b).*

In Theorem 1 take $T=(a, b)$, $X=BV_0$ so that

$$|\gamma \Delta f| = |\Delta \gamma f| \leqq \int_a^b |d\gamma f| \leqq \|f\| \cdot \|\gamma\|$$

and thus $\|\Delta f\| \leqq \|f\|$.

We should like to point out another proof of this theorem. In order to put it in the notation of the previous theorems we use t in place of Δ and put $F(t)=tf$. Then $\gamma F(t)=t\gamma f$ is in the space $m(T)$ for every γ in $\overline{Y}$; hence Theorem 12 is a corollary of Theorem 2 applied to $F(t)$.

This latter point of view will save some time for it makes the following three theorems corollaries of Theorems 3, 7, and 10, respectively.

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* This restriction is merely for simplicity of notation.

THEOREM 13. *If f_P on (a, b) to $\overline{Y}$ is such that $f_P y$ is in BV_0 for each y in Y, then f_P is of bounded variation on (a, b).*

THEOREM 14. *Let U be a space in which there is a notion of null set satisfying condition (N), and let Y be a separable Banach space. If $f_{u,P}$ on $U(a, b)$ to $\overline{Y}$ is such that for each y in Y there is a null set $U_y \subset U$ and an M_y such that*

$$\int_a^b |d_P f_{u,P} y| \leqq M_y, \qquad u \in U - U_y,$$

then there is a null set $U_0 \subset U$ such that $f_{u,P}$ is of bounded variation in P on (a, b) uniformly with respect to u in $U - U_0$.

The v in Theorem 7 plays the role here of t or Δ, and Theorem 7 applied to the function $F_{u,v} = vf_{u,P}$ gives the present theorem. Similarly Theorem 10 gives the theorem:

THEOREM 15. *Let U be a set (u) in which there is a notion of null set satisfying condition (N). Let $\overline{Y}$ be an f.s. space with determining manifold Γ. If $f(u, P)$ on $U(a, b)$ to Y is such that for each γ in Γ there is a null set $U_\gamma \subset U$ and an M_γ such that*

$$\int_a^b |d_P \gamma f(u, P)| \leqq M_\gamma, \qquad u \in U - U_\gamma,$$

then there is a null set $U_0 \subset U$ such that $f(u, P)$ is of bounded variation in P on (a, b) uniformly with respect to u in $U - U_0$.

THEOREM 16. *If $f(p)$ on (a, b) to Y is such that the Riemann integral*

$$\int_a^b \phi(p) d\gamma f(p)$$

exists for every continuous function ϕ and every γ in $\overline{Y}$, then the Riemann integral

$$\int_a^b \phi(p) df(p)$$

exists for every continuous function ϕ.

In view of Theorems 11 and 12 it is sufficient to show that the existence of the Riemann integral $\int_a^b \phi(p) d\psi(p)$ of an arbitrary continuous function with respect to the real function ψ implies* $\int_a^b |d\psi| < \infty$.

* This fact is probably well known. It is stated without reference in the introduction of a paper of Pollard [27]. Not knowing where it is proved and because the above proof is another simple application of the principle of uniform boundedness we give the details here.

If ψ is not of bounded variation on (a, b), a sequence π_n of partitions of (a, b) with norm approaching zero can be formed in such a way that each partition π_n is composed of intervals δ_j, δ_j' with

$$\left| \sum_j \delta_j \psi \right| > n.$$

From the principle of uniform boundedness, however, there is a constant M independent of n such that for an arbitrary continuous function $\phi(p)$ we have the inequality

$$\left| \sum_j \phi(\tau_j)\delta_j\psi + \sum_j \phi(\tau_j')\delta_j'\psi \right| \leqq M \sup_{a \leqq p \leqq b} | \phi(p) |,$$

where τ_j (τ_j') is a point in δ_j (δ_j'). For any $n > M$ let ϕ be the continuous function which vanishes on δ_j' and on δ_j has for its graph an isosceles triangle with base δ_j and height one. Then if τ_j is the center of δ_j, we have

$$n < \left| \sum_j \delta_j \psi \right| = \left| \sum_j \phi(\tau_j)\delta_j\psi \right| \leqq M,$$

a contradiction.

In a similar manner it is possible to demonstrate the following theorem:

THEOREM 17. *If f_p on (a, b) to $\overline{Y}$ is such that the Riemann integral*

$$\int_a^b \phi(p) df_p(y)$$

exists for every continuous function ϕ and every y in Y, then the Riemann integral

$$\int_a^b \phi(p) df_p$$

also exists for every continuous function ϕ.

For simplicity of statement we have avoided a generalization which might be made in Theorems 2 to 15 inclusive (except 8 and 11). In all of these theorems the set of $\gamma[y]$ for which $\gamma f(t) [f_t y]$ is bounded, or essentially bounded, need not be assumed to be the whole of $\Gamma[Y]$ but merely a set of second category in $\Gamma[Y]$. We have so far only used Theorem 1 in the case where $X = M(T)$ or $M^*(T)$ and each of these spaces is a special case of a Banach space X where the following condition holds. *If $\|\phi_p\| \leqq M$ for $p = 1, 2, \cdots$ and $\phi_p(t) \rightarrow \phi(t)$ for every t in T, then ϕ is in X, and $\|\phi\| \leqq M$.* For Banach spaces X satisfying this condition it is readily shown that if

$\gamma f(.)$ is in X for every γ in a set of the second category in Γ, then $\gamma f(.)$ is in X for every γ in Γ. For the set $\Gamma_0 \subset \Gamma$, where $\gamma f(.)$ is in X, is a linear set and contains a sphere in Γ since it is a second category sum of closed sets $\Gamma_n = \Gamma[\|\gamma f(.)\|_X \leqq n]$. Thus in the case of a space X satisfying the above condition, Theorem 1 can be worded as follows:

THEOREM 1′. *Let X be a Banach space satisfying, besides the conditions* (1), (2), (3), *the condition stated in the preceding paragraph. If the function $f(t)$ on an arbitrary set T to a Banach space Y is such that $\gamma f(.)$ is in X for every γ in a set of second category in the closed linear manifold $\Gamma \subset \overline{Y}$, then f is in $\mathfrak{X}[Y, \Gamma]$ and $\gamma f(.)$ is a continuous operation from Γ to X.*

From this statement of Theorem 1 the generalization mentioned above is obvious.

1.2. **Adjoint operations.** It has been proved by von Neumann [21], Stone ([34], p. 61, Theorem 2.26), Tamarkin ([34], p. iv), and Stone and Tamarkin [35] that an additive operation on Hilbert space to Hilbert space is continuous providing its adjoint is everywhere defined. The corresponding theorem for Banach spaces, together with a number of similar theorems, can be obtained from Theorem 1 by specializing the range T, the function f, and the space X. As the reader will see, these theorems are all of a general type asserting that a limiting process when existing in a weak sense will also exist in a strong sense, and he may therefore expect to find them in Chapter II where such phenomena are discussed. Since the operations involved are additive functions and for such functions continuity is equivalent to boundedness, and since the proofs are entirely characteristic of the preceding proofs, we prefer to group these theorems with those on uniform boundedness.

In this section Y and Z are arbitrary Banach spaces.

By a *determining manifold* in $\overline{Y}$ will be meant a closed linear manifold Γ in $\overline{Y}$ such that

$$\sup_{\|\gamma\|=1} \gamma y = \|y\|, \qquad y \in Y.$$

We shall use the letter Γ for a determining manifold in $\overline{Y}$. It will not be assumed that Γ is separable unless it is so stated. The symbol Γ^* will be used for a set of elements in $\overline{Y}$ (not necessarily in Γ) such that for every γ in Γ there is a sequence γ_n^* of finite linear combinations of elements of Γ^* such that $\gamma_n^* y \to \gamma y$ for every y in Y. The symbols γ, γ^*, μ, with or without subscripts or superscripts, will always stand for points in Γ, Γ^*, $\overline{Z}$, respectively.

We shall be considering functions $y = f(z)$ on a set $D(f)$ (domain of f) dense in Z and with values in Y. The *adjoint* $\bar{f}$ of f is a function on a subset of $\overline{Y}$

with values in $\bar{Z}$. The domain of definition $D(\bar{f})$ of $\bar{f}$ consists of those $\bar{y}$ in $\bar{Y}$ for which there exists a μ such that

$$\bar{y}f(z) = \mu z, \qquad z \in D(f).$$

Since $D(f)$ is dense in Z, μ is unique. The function $\bar{f}$ is then defined by the equation

$$\bar{f}(\bar{y}) = \mu, \qquad \bar{y} \in D(\bar{f}).$$

THEOREM 18. *If the domain $D(f)$ of $y=f(z)$ is dense in Z, and if the adjoint of f is defined for every γ in Γ, then $D(\bar{f})=\bar{Y}$, $\bar{f}$ is continuous, and the domain $D(f)$ may be extended to the whole of Z in such a way that the extended function $f(z)$ is a bounded linear operator with norm the same as that of $\bar{f}$.*

First note that Theorem 2 holds if $\bar{Y}$ is replaced by a determining manifold Γ in $\bar{Y}$. Now in Theorem 2 take T as the set of all z in $D(f)$ with $\|z\| \leqq 1$. By Theorem 2

$$\|f(z)\| \leqq M, \qquad z \in D(f), \|z\| \leqq 1.$$

The rest of the proof is obvious, and we leave it to the reader.

THEOREM 19. *If the adjoint of $y=f(z)$ on Z to Y is defined for every γ^* in Γ^*, then f and $\bar{f}$ are bounded linear operators with the same bound.*

It is a well known corollary of Theorem 3 that if a sequence $\{\mu_n\}$ of linear functionals converges for every z, then the limit is a linear functional. Thus the adjoint $\bar{f}$ is defined for every γ in Γ; and this theorem is a corollary of the preceding one.

THEOREM 20. *If the additive function $y=f(z)$ on Z to Y has the property that $\gamma^*f(z)$ is continuous for every γ^* in Γ^*, then f is continuous.*

This is merely a restatement of Theorem 19, and it is in this form that we prefer to state the further theorems of this type.

We note in passing that for Banach spaces Y which are equivalent to their own conjugates it is possible to define the notion of a *symmetric transformation* and obtain a corollary to Theorem 19 which states that *if an additive symmetric transformation is defined everywhere, then it is continuous.* For example a function $f(y)$ on $D(f) \subset Y$ to Y (where $Y=\bar{Y}$) might be defined as symmetric if it obeys the law

$$y'f(y) = yf(y'), \qquad y, y' \in D(f).$$

THEOREM 21. *If the additive function f_z on Z to $\bar{Y}$ has the property that $f_z y$ is continuous in z for each y in a fundamental set in Y, then f_z is continuous.*

This follows from Theorem 20 by replacing Y by $\overline{Y}$, Γ by Y, and Γ^* by a fundamental set in Y.

THEOREM 22. *If f and f_n, $(n=1, 2, \cdots)$, are additive functions on Z to Y such that $\gamma^* f_n(z)$ is continuous for each γ^* in Γ^* and every integer n, and if*

$$\gamma^* f_n(z) \to \gamma^* f(z), \qquad \gamma^* \in \Gamma^*, z \in Z,$$

then f is continuous.

This follows from Theorems 3 and 20. We leave the details of this theorem, as well as the next, to the reader.

THEOREM 23. *If f_z and f_z^n, $(n=1, 2, \cdots)$, are additive functions on Z to $\overline{Y}$ such that $f_z^n(y)$ is continuous in z for every y in a fundamental set in Y and every integer n, and if*

$$\lim_n f_z^n(y) = f_z(y),$$

for z in Z and y in a fundamental set in Y, then f_z is continuous.

THEOREM 24. *Let S be an arbitrary set (s) of elements, and let $f(z, s)$ on ZS to Y be such that*

(i) *for each γ in Γ and z in Z, $\gamma f(z, s)$ is bounded on S;*

(ii) *for each γ in Γ and s in S, $\gamma f(z, s)$ is a continuous linear functional in z.*

Then $f(z, s)$ is continuous and linear in z uniformly with respect to s; that is, there is a constant M such that

$$\sup_s \|f(z, s)\| \leqq M\|z\|, \qquad z \in Z.$$

Let $T=ZS$, and take X as the space of all real functions $\mu(z, s)$ on T which satisfy the following conditions:

(a) For each z, $\mu(z, s)$ is bounded on S.

(b) For each s, $\mu(z, s)$ is a continuous linear functional on Z.

Then by Theorem 3

$$\sup_s \|\mu(., s)\| = \sup_s \sup_{\|z\|=1} | \mu(z, s) | < \infty .$$

This constant is taken as the norm of a point in X. Theorem 24 is then a corollary of Theorem 1.

THEOREM 25. *Let S be an arbitrary set of elements (s) and $f_{z,s}$ on ZS to $\overline{Y}$ be such that*

(i) *for y in Y and z in Z, $f_{z,s}(y)$ is bounded on S;*

(ii) *for y in Y and s in S, $f_{z,s}(y)$ is continuous and linear in z.*

Then $f_{z,s}$ is linear and continuous in z uniformly with respect to s, that is, there is a constant M such that

$$\sup_s \|f_{z,s}\| \leqq M\|z\|, \qquad z \in Z.$$

This follows from Theorem 24 by replacing Y by $\bar{Y}$ and Γ by Y.

THEOREM 26. *Let S be an arbitrary set* (s) *in which there is defined a notion of null set satisfying condition* (N). *Let Z be separable and $f(z, s)$ on ZS to Y such that*

(i) *for each γ in Γ and z in Z there is a constant $M(\gamma, z)$ and a null set $S(\gamma, z)$ such that $|\gamma f(z, s)| \leqq M(\gamma, z)$ on $S-S(\gamma, z)$;*

(ii) *for each γ in Γ and s in S, $\gamma f(z, s)$ is continuous and linear in z.*

Then there is a constant M such that

$$\text{ess. sup.}_s \sup_{\|z\|=1} |\gamma f(z, s)| \leqq M\|\gamma\|;$$

and if Γ is separable, there is a null set S_0 such that

$$\|f(z, s)\| \leqq M\|z\| \qquad s \in S - S_0,\ z \in Z.$$

Take $T = ZS$ and X as the space of all real functions on T of the form $\mu(z, s)$, where

(a) for each s, $\mu(z, s)$ is continuous and linear in z;

(b) for each z, $\mu(z, s)$ is essentially bounded in s.

Then by Theorem 5

$$\text{ess. sup.}_s \|\mu(.\,, s)\| = \text{ess. sup.}_s \sup_{\|z\|=1} |\mu(z, s)| < \infty,$$

and this constant is taken as the norm in X. The first conclusion follows from Theorem 1. If $\{\gamma_j\}$ is dense in Γ, and if S_j the set in S where

$$\sup_{\|z\|=1} |\gamma_j f(z, s)| > M\|\gamma_j\|,$$

then the second conclusion follows by taking $S_0 = \sum_{j=1}^{\infty} S_j$.

THEOREM 27. *Let S and Z be as in Theorem 26, and $f_{z,s}$ on ZS to $\bar{Y}$ be such that*

(i) *for each y in Y and z in Z there is a constant $M(y, z)$ and a null set $S(y, z)$ such that $|f_{z,s}(y)| \leqq M(y, z)$ on $S-S(y, z)$;*

(ii) *for each y in Y and s in S, $f_{z,s}(y)$ is continuous and linear in z.*

Then there is a constant M such that

$$\text{ess. sup.}_s \sup_{\|z\|=1} |f_{z,s}(y)| \leqq M\|y\|.$$

If Y is separable, then there is a null set S_0 such that

$$\|f_{z,s}\| \leqq M\|z\| \qquad \text{on} \qquad S - S_0.$$

This follows from Theorem 26 by replacing Y by $\overline{Y}$ and Γ by Y.

The theorems of this section may be conveniently applied to the theory of multilinear forms. One has, for example, the following theorem:

THEOREM 28. *Let Y, Z, Z' be Banach spaces, and $f(z, z')$ on ZZ' to Y be additive in each argument and such that for each γ^* in Γ^*, $\gamma^*f(z, z')$ is continuous in each of the variables z, z' separately. Then there is a constant M such that*

$$\|f(z, z')\| \leqq M\|z\|\cdot\|z'\|.$$

From the continuity of $\gamma^*f(z, z')$ in z and z' separately follows that of $\gamma f(z, z')$ for any γ in Γ. The desired conclusion follows from Theorem 24 by taking S as the unit sphere in Z'. A corollary is the following:

THEOREM 29. *Let $f_{z,z'}$ on ZZ' to $\overline{Y}$ be additive in each argument and such that for each y_0 in a fundamental set in Y, $f_{z,z'}(y_0)$ is continuous in z, z' separately. Then there is a constant M such that*

$$\|f_{z,z'}\| \leqq M\|z\|\cdot\|z'\|.$$

We shall leave further applications of this sort to the reader.

1.3. **Concluding remarks.** Although we shall not have much occasion to use the fact, it might be pointed out that for certain Banach spaces X, the corresponding space $\mathfrak{X}[Y, \Gamma]$ (where Γ is a determining manifold in Y) is itself a Banach space. Suppose the Banach space X satisfies, besides the conditions (1), (2), (3), the further condition:

(A) *If $\phi_n \to \phi$, then $\phi_n(t) \to \phi(t)$ for each t in T.*

It follows from this that $\phi(t) = 0$ on T providing $\phi = 0$, and also that for fixed t, $\nu_t\phi = \phi(t)$ is a linear functional on X. Thus

$$|\phi(t)| \leqq \|\nu_t\|\cdot\|\phi\|, \qquad \phi \in X.$$

This shows that if s is an arbitrary parameter and

$$\lim_{m,n} \|\phi_n^s - \phi_m^s\| = 0$$

uniformly in s, then for each t

$$|\phi_m^s(t) - \phi_n^s(t)| \leqq \|\nu_t\|\cdot\|\phi_m^s - \phi_n^s\| \to 0$$

uniformly in s. Now suppose Γ is a determining manifold in $\overline{Y}$ and $\{f_m\}$ is a Cauchy sequence of points in $\mathfrak{X}[Y, \Gamma]$. Then

$$\lim_{m,n} \sup_{\|\gamma\|=1} \|\gamma f_m(.) - \gamma f_n(.)\| = 0,$$

and thus

$$\lim_{m,n} \sup_{\|\gamma\|=1} |\gamma f_m(t) - \gamma f_n(t)| = 0, \qquad \text{for each } t,$$

so that the sequence f_m defines uniquely a function $f(t)$ on T to Y such that

$$f_n(t) \to f(t), \qquad t \in T.$$

It is readily shown that f is in $\mathfrak{X}[Y, \Gamma]$ and that $\|f_n - f\| \to 0$ where the norm is taken in the space $\mathfrak{X}$. Thus $\mathfrak{X}[Y, \Gamma]$ is complete. This, together with certain other obvious facts, shows that $\mathfrak{X}[Y, \Gamma]$ is a Banach space.

Now let $U(\gamma)$ be an arbitrary continuous linear operation from Γ to X, and suppose that $Y = \overline{\overline{Y}}$. Then $\nu_t U(\gamma)$, being continuous in γ, is representable as $\nu_t U(\gamma) = \bar{\gamma}(\gamma)$, where $\bar{\gamma}$ is a point of $\overline{\Gamma}$. By the Hahn-Banach theorem on the extension of linear functionals, there is a point $\bar{\bar{y}}$ in $\overline{\overline{Y}}$ such that $\bar{\bar{y}}(\gamma) = \bar{\gamma}(\gamma)$ for γ in Γ; and since $Y = \overline{\overline{Y}}$, we have for each t in T a point $f(t)$ in Y such that $\nu_t U(\gamma) = \gamma f(t)$ and thus $U(\gamma) = \gamma f(.)$. This shows that for spaces Y equal to their second conjugates the operation of Theorem 1 is the general linear operator from Γ to X. To summarize we state the following theorem:

THEOREM 30. *Let X be a Banach space which satisfies, besides the conditions* (1), (2), (3), *the further condition* (A). *Let Γ be a determining manifold in $\overline{Y}$. Then the space $\mathfrak{X}[Y, \Gamma]$ with the norm $\|f\|$ of Theorem* 1 *is a Banach space. If in addition $Y = \overline{\overline{Y}}$, then every continuous linear operator $U(\gamma)$ on Γ to X is expressible in the form*

$$U(\gamma) = \gamma f(.),$$

where f is a point of $\mathfrak{X}[Y, \Gamma]$.

The condition (A) will not be assumed at any other place in this paper.

CHAPTER II

2.0. **Uniform limiting processes.** Most of the applications of Theorem 1 that have been given thus far have been of the general type asserting that boundedness in a weak sense implies boundedness in a strong sense. The applications of the present chapter are to cases where certain limiting processes, when existing in the weak sense, also exist in the strong sense. This is closely related to, and in many cases synonymous with, the statement that the operation of Theorem 1 is not only continuous but completely continuous. This is the case when X is l or L, $\Gamma = \overline{Y}$, and Y is a separable space equal to its second conjugate. Cases other than those in this chapter where a weak limiting process implies a strong one will be found in Chapter IV.

Before proceeding to the results of this chapter we desire to point out the connection between Theorem 32 and a theorem of Orlicz-Banach. Orlicz [23] has shown that for weakly complete spaces the unconditional (that is, absolute in this case) convergence of $\sum_{n=1}^{\infty} \gamma y_n$ for every γ in $\overline{Y}$ implies the unconditional convergence of $\sum y_n$. Banach ([1], p. 240) states this theorem in a more general form, namely: If all partial sums of $\sum y_n$ converge weakly to an element in Y, then $\sum y_n$ converges unconditionally. In this form it is not assumed that Y is weakly complete, and the proof can be carried out in a fashion similar to that of Orlicz.*

In view of the known conditions for conditional compactness in l (see, for example, [5], Theorem 2) we can give the following proof of the Orlicz-Banach theorem. Let S be the unit sphere in $\overline{Y}$ and $U(\gamma)=\{\gamma y_n\}$ on $\overline{Y}$ to l which, by Theorem 1, is continuous. It is also completely continuous, for from any sequence in S there is a subsequence γ_i converging on the closed linear manifold determined by y_n, $(n=1, 2, \cdots)$. Since Theorem 1 shows that the sequence $\{U(\gamma_i)\}$ is bounded, to show it convergent weakly and hence in l, it is sufficient to show that $fU(\gamma_i)$ converges for every f in a fundamental set in $\bar{l}$. Such a fundamental set is the set of characteristic functions of sets σ of integers. For such an $f=f_\sigma$,

$$f_\sigma U(\gamma_i) = \sum_{n\epsilon\sigma} \gamma_i y_n = \gamma_i y_\sigma,$$

and this converges since y_σ is in the closed linear manifold determined by y_n. Thus $U(S)$ is conditionally compact; therefore

$$\lim_N \sum_{n=N}^{\infty} | \gamma y_n | = 0$$

uniformly on S, which shows that $\sum_{n\epsilon\sigma} y_n = y_\sigma$ for every set σ of integers.

Theorem 32 to follow is a generalization of the Orlicz-Banach theorem, but the proof even in the case considered by these authors is different from that of Orlicz.

We shall now assume that X is a Banach space of numerical functions $\phi(p)$ on a range P satisfying, besides the conditions (1), (2), and (3) (with T replaced by P), the further condition:

(4) *If ϕ_i is in X, $(i=1, 2, \cdots)$, and $\phi_i(p)\rightarrow\phi(p)$ for p in P, and if $\nu\phi_i$ converges for every ν in $\overline{X}$, then ϕ is in X and $\phi_i\rightarrow\phi$ in X.*

It is evident that l has this property, for weak and strong convergence in l are equivalent. Another space in which we shall be interested, which satisfies conditions (1) to (4), is the space $L(E)=L(E, \alpha)$. In this symbolism E is an abstract set, and α a completely additive measure function defined on a σ-field $\mathfrak{a}(E)$ of "measurable" subsets of E. The space $L(E)$ is then the space

* Pettis has also given a proof of this theorem; see [24].

of numerical functions on E summable on E with respect to the total variation $|\alpha|$ of α, and the norm of a point ϕ in $L(E)$ is

$$\|\phi\| = \int_E |\phi(p)| \, d|\alpha|.$$

The space $L(E)$ enjoys the property (4), for if $\nu\phi_i$ converges for every ν in $\overline{L(E)}$, we have in particular

$$\lim_i \int_e \phi_i(p) d\alpha$$

existing for every measurable subset e of E. Thus by a theorem of Vitali-Hahn-Saks [31] the integrals are equi-absolutely continuous. It follows immediately that ϕ is in $L(E)$ and $\phi_i \to \phi$ in $L(E)$.

In what follows in this chapter $P=(p)$ and $T=(t)$ are arbitrary sets, X is a Banach space of numerical functions $\phi(p)$ on P which satisfies conditions (1) to (4), and F is a fundamental set in $\overline{X}$. The space is an arbitrary Banach space, Γ is an arbitrary closed linear manifold in $\overline{Y}$, $y(p, t)$ is a function on PT to Y, and Y_0 is the closed linear manifold in Y determined by $y(P, T)$. The symbol $M(X)$ will be used for the space of numerical functions $\phi(p, t)$ in X for each t in T with $\|\phi\| = \sup_t \|\phi(.,t)\|_X < \infty$. The following assumptions will sometimes be made:

I. *For every p in P the set $y(p, T)$ is conditionally compact.*

II. *Either Y_0 is separable, or every bounded sequence in Γ contains a subsequence γ_i such that $\gamma_i y_0$ converges for every y_0 in Y_0.*

Note that the first of the two alternative assumptions in II implies the second, and that I implies II in case the range P is denumerable.

In the following theorem the domain $D(U)$ of U is the set of all μ in $\overline{Y}$ for which $U(\mu) = \mu y(p, t)$ is in $M(X)$.

THEOREM 31. *Assume* I *and* II, *and that*

(i) $\Gamma \subset D(U)$, *and*

(ii) *for f in F there is a $y_f(t)$ on T to Y_0 with $y_f(t)$ conditionally compact and*

$$f\gamma y(.,t) = \gamma y_f(t), \qquad \gamma \in \Gamma,\ t \in T.$$

It follows that

(iii) *if $\{\gamma_i\}$ is a bounded sequence in Γ, μ a point of $\overline{Y}$, and $\gamma_i y_0 \to \mu y_0$ for y_0 in Y_0, then μ is in $D(U)$ and $U(\gamma_i)$ approaches $U(\mu)$ in $M(X)$;*

(iv) *in case $\Gamma = \overline{Y}$ and $\gamma_i y_0 \to 0$ for y_0 in Y_0, then $U(\gamma_i) \to 0$ in $M(X)$; and*

(v) *$U(\gamma)$ is completely continuous on Γ to $M(X)$.*

To prove (iii) we have by Theorem 1

$$\|U(\gamma_i)\| \leqq \|U\| \sup_i \|\gamma_i\|,$$

so that

$$\sup_t \|\gamma_i y(.,t)\|_X \leqq M.$$

For f in F, $f\gamma_i y(.,t)=\gamma_i y_f(t)$, and this sequence converges for each t. Thus since $\gamma_i y(.,t)$ is a bounded sequence in X, it is weakly convergent. Also $\gamma_i y(p,t)\rightarrow\mu y(p,t)$ for every p and t. Hence for each t, $\mu y(.,t)$ is in X and $\gamma_i y(.,t)\rightarrow\mu y(.,t)$ in X. Thus

$$\|\mu y(.,t)\|_X = \lim_i \|\gamma_i y(.,t)\|_X \leqq M,$$

so that $U(\mu)$ is in $M(X)$. If $U(\gamma_i)$ does not approach $U(\mu)$ in $M(X)$, there are sequences $i_q\rightarrow\infty$ and t_q, and an $\epsilon>0$ such that

$$0 < \epsilon < \|\lambda_q y(.,t_q)\|_X = \|\phi_q\|_X \leqq 2M,$$

where $\lambda_q=\gamma_{i_q}-\mu$ and $\phi_q=\lambda_q y(.,t_q)$ is a point in X.

To obtain the contradiction we shall show that

(a) $\phi_q(p)\rightarrow 0$ for each p in P, and

(b) $f\phi_q$ converges for every f in $\overline{X}$.

Since $V_p\equiv y(p,T)$ is conditionally compact it is totally bounded and is thus covered by a finite number of spheres $K(y_i,\delta)$, $(i=1, 2, \cdots, n_\delta)$, with centers y_i in v_p and radii δ. There is a q_δ such that

$$|\lambda_q y_i| \leqq \delta, \qquad i = 1, 2, \cdots, n_\delta;\ q \geqq q_\delta,$$

and for each y in V_p there is an i such that $\|y-y_i\|_Y<\delta$. Thus for any y in V_p

$$\begin{aligned} |\lambda_q y| &\leqq |\lambda_q(y-y_i)| + |\lambda_q(y_i)| \\ &\leqq \delta\left[\sup_q \|\lambda_q\| + 1\right], \qquad q \geqq q_\delta. \end{aligned}$$

This shows that $\lambda_q y\rightarrow 0$ uniformly on V_p; hence (a) is true. Since $\|\phi_q\|_X\leqq 2M$, it suffices, in proving (b), to show that $f\phi_q$ converges for every f in F. Now from (ii), and the fact that $\gamma_i y(.,t)\rightarrow\mu y(.,t)$, we have

$$f\phi_q = f\lambda_q y(.,t_q) = \lambda_q y_f(t_q),$$

and since $y_f(T)$ is conditionally compact, it follows as above that $f\phi_q\rightarrow 0$. This completes the proof of (iii).

To prove (iv) note that while $\|\gamma_i\|$ may not be bounded, the sequence $\|\gamma_i\|_0=\sup_{\|y_0\|=1}|\gamma_i y_0|$ (where y_0 is in Y_0) is bounded. Thus by a theorem of Hahn-Banach ([1], p. 27) on the extension of linear functionals, there is a sequence $\{\lambda_i\}$ in $\overline{Y}$ such that $\{\lambda_i\}$ is bounded in $\overline{Y}$ and λ_i coincides with γ_i on Y_0. Conclusion (iv) is then a corollary of (iii). In view of II, for every bounded sequence in Γ there is a subsequence γ_i and a functional μ such that $\gamma_i y \to \mu y$ on Y_0. Thus (v) also follows from (iii).

Theorem 31 enables us to state eight theorems some of which will be of use later in the theory of integration. These theorems deal with the following Banach spaces (where $T=(t)$ is an arbitrary set of elements):

E_1 is the space composed of sequences of numerical functions $\phi_n(t)$ such that $\sup_t \sum_{n=1}^{\infty} |\phi_n(t)| < \infty$. The norm is then defined as

$$\|\phi\| = \sup_t \sum_{n=1}^{\infty} |\phi_n(t)|.$$

E_2 is the subset of E_1 for which

$$\lim_{N=\infty} \sup_t \sum_{n=N}^{\infty} |\phi_n(t)| = 0.$$

The norm in E_2 is the same as that in E_1.

E_3: Suppose that in T there is a notion of null set satisfying condition (N). Then E_3 is the set of sequences of numerical functions $\phi_n(t)$ such that $\text{ess. sup.}_t \sum_{n=1}^{\infty} |\phi_n(t)| < \infty$. The norm is

$$\|\phi\| = \underset{t}{\text{ess. sup.}} \sum_{n=1}^{\infty} |\phi_n(t)|.$$

E_4 is the subset of E_3 for which

$$\lim_{N=\infty} \underset{t}{\text{ess. sup.}} \sum_{n=N}^{\infty} |\phi_n(t)| = 0.$$

The norm in E_4 is the same as in E_3.

Let E be an abstract set, and α a completely additive numerical measure function defined on a σ-field $\mathfrak{a}(E)$ of "measurable" subsets of E. Let $|\alpha|(e)$ be the total variation of α on e, and suppose that $|\alpha|(E) < \infty$.

E_5 is the space of numerical functions $\phi(p, t)$ on ET such that, for each t in T, $\phi(p, t)$ is summable relative to $|\alpha|$ on E and such that $\sup_t \int_E |\phi(p, t)| d|\alpha| < \infty$. The norm is then

$$\|\phi\| = \sup_t \int_E |\phi(p, t)| d|\alpha|.$$

E_6 is the subspace of E_5 for which

$$\lim_{|\alpha|(e)=0} \sup_t \int_e |\phi(p, t)| d|\alpha| = 0.$$

The norm in E_6 is the same as for E_5.

E_7: In case there is a notion of null set in T satisfying condition (N), then E_7 is the space of numerical functions on ET such that for almost all t in T, $\phi(p, t)$ is summable with respect to α on E and

$$\|\phi\| = \text{ess. sup.}_{t} \int_E |\phi(p, t)| \, d|\alpha| < \infty .$$

E_8 is the subspace of E_7 consisting of all functions for which

$$\lim_{|\alpha|(e)=0} \text{ess. sup.}_{t} \int_E |\phi(p, t)| \, d|\alpha| = 0.$$

The norm in E_8 is the same as in E_7.

In case the set T has but a single element, the first four of these spaces reduce to l while the last four reduce to $L = L(E, \alpha)$.

THEOREM 32. *Let* $\{y_n(t)\}$ *be a sequence of functions on* T *to* Y *such that*

(i) *for every set* σ *of integers there is a* $y_\sigma(t)$ *on* T *to* Y_0 *such that*

$$\sum_{n \in \sigma} \gamma y_n(t) = \gamma y_\sigma(t), \qquad t \in T, \gamma \in \Gamma;$$

(ii) *for every* γ *in* Γ, $U(\gamma) \equiv \{\gamma y_n(t)\}$ *is in* E_1; *and*

(iii) *for every* n *and* σ *the sets* $y_n(T)$, $y_\sigma(T)$ *are conditionally compact.*

Then it follows that

(iv) *the transformation* $U(\gamma)$ *on* Γ *to* E_1 *is completely continuous*;

(v) *for each* t *in* T

$$\lim_{N=\infty} \sum_{n \geqq N} |\gamma y_n(t)| = 0$$

uniformly with respect to $\|\gamma\| \leqq 1$;

(vi) *the set* Γ *of linear functionals for which it is assumed that* $U(\gamma)$ *is defined can be extended to include any* γ *in* $\overline{Y}$ *for which there exists a sequence* γ_i *in* Γ *with* $\|\gamma_i\|$ *bounded and* $\gamma_i y_n(t) \to \gamma y_n(t)$ *for every* n *and* t. *If* γ_i *and* γ *are such functionals, then*

$$\lim_{j} \sup_{t} \sum_{n=1}^{\infty} |\gamma_j y_n(t) - \gamma y_n(t)| = 0;$$

(vii) *in case* $\Gamma = \overline{Y}$ *and* $\gamma_j y \to 0$ *for* y *in* Y_0, *then*

$$\lim_{j=\infty} \sup_{t} \sum_{n=1}^{\infty} |\gamma_j y_n(t)| = 0;$$

(viii) *in case* Γ *is a determining manifold in* $\overline{Y}$, *then for every set* σ *of integers the series* $\sum_{n \in \sigma} y_n(t)$ *converges on* T *to* $y_\sigma(t)$.

Most of these conclusions follow immediately from the preceding theorem. Here the set P is replaced by the set of integers so that condition II is automatically satisfied. Conclusion (v) follows from (iv) by applying the conditions for compactness in l, and conclusion (viii) follows immediately from (v).

It should be pointed out that if $\Gamma = \overline{Y}$, and Y is weakly complete, then in hypothesis (i) the existence of $y_\sigma(t)$ is implied by the convergence of $\sum_{n\epsilon\sigma}\gamma y_n(t)$. Also if T has but a finite number of points, then (iii) is automatically satisfied. *Thus if $\overline{Y} = \Gamma$, if Y is weakly complete, and if $X = l$, the transformation of Theorem* 1 *is completely continuous*. Similar remarks hold for the following theorems:

THEOREM 33. *Under the hypothesis of the preceding theorem, except that now it is assumed that $U(\gamma)$ is in E_2 for every γ in Γ, the conclusions* (v) *and* (viii) *can be strengthened to the following statements*:

(v′) *uniformly with respect to $\|\gamma\| \leqq 1$,*

$$\lim_N \sup_t \sum_N^\infty | \gamma y_n(t) | = 0;$$

(viii′) *in case Γ is a determining manifold in $\overline{Y}$ then, for every set σ of integers, the series $\sum_{n\epsilon\sigma} y_n(t)$ converges uniformly on T to $y_\sigma(t)$.*

If (v′) were not true, there would exist sequences $N_i \rightarrow \infty$, t_i, and γ_i, an $\epsilon > 0$, and a functional γ (perhaps not in Γ) such that

$$\|\gamma_i\| \leqq 1, \quad \gamma_i y_0 \rightarrow \gamma y_0, \quad y_0 \epsilon Y_0; \quad 0 < \epsilon < \sum_{N_i}^\infty | \gamma_i y_n(t_i) |, \qquad i = 1, 2, \cdots .$$

By (vi) of the preceding theorem $U(\gamma_i) \rightarrow U(\gamma)$ in E_1, and, since E_2 is a closed linear manifold in E_1, $U(\gamma)$ must be in E_2. Thus

$$\limsup_i \sum_{N\cdot}^\infty | \gamma_i y_n(t_i) | \leqq \lim_i \sup_t \sum_{n=1}^\infty | \gamma_i y_n(t) - \gamma y_n(t) |$$

$$= \lim_i \sup_t \sum_{N_i}^\infty | \gamma y_n(t) | = 0,$$

which is the desired contradiction. Conclusion (viii′) is a corollary of (v′).

THEOREM 34. *Assume* I *and* II *and that $U(\gamma) = \gamma y(p, t)$ on Γ to E_5 is such that*

(i) *for every e in $\mathfrak{a}(E)$ there is a $y_e(t)$ on T to Y_0 with $y_e(T)$ conditionally compact and*

$$\gamma y_e(t) = \int_e \gamma y(p, t) d\alpha, \qquad \gamma \epsilon \Gamma.$$

Then

(ii) *if* $\{\gamma_i\}$ *is a bounded sequence in* Γ, μ *a point of* $\overline{Y}$, *and* $\gamma_i y_0 \rightarrow \mu y_0$ *for* y_0 *in* Y_0, *then* $U(\mu)$ *is in* E_5 *and*

$$\lim_i \sup_t \int_E |\gamma_i y(p, t) - \mu y(p, t)| d|\alpha| \rightarrow 0;$$

(iii) *for each* t

$$\lim_{|\alpha|(e)=0} \int_e |\gamma y(p, t)| d|\alpha| = 0$$

uniformly with respect to $\|\gamma\| \leqq 1$;

(iv) U *is completely continuous*;

(v) *in case* $E=(0, 1)$ *and* α *is Lebesgue measure, then for each* t

$$\lim_{h=0} \int_0^{1-h} |\gamma y(p + h, t) - \gamma y(p, t)| dp = 0$$

uniformly for $\|\gamma\| \leqq 1$.

This follows from Theorem 31. Here F is the set of characteristic functions of measurable sets e. In view of II, conclusion (iii) (which however is not as strong as (iv)) follows from (ii), and (v) is a known condition for compactness in L [37].

THEOREM 35. *If, in addition to the assumptions of the preceding theorem, we assume that for each* γ *in* Γ

$$\lim_{|\alpha|(e)=0} \sup_t \int_e |\gamma y(p, t)| d|\alpha| = 0,$$

then this limit exists uniformly with respect to $\|\gamma\| \leqq 1$.

If the conclusion were not true, there would exist sequences t_i, γ_i, e_i (with $|\alpha|(e_i) \rightarrow 0$), an $\epsilon > 0$, and a γ (perhaps not in Γ) such that

$$\|\gamma_i\| \leqq 1,$$

$$\gamma_i y_0 \rightarrow \gamma y_0, \qquad y_0 \in Y_0,$$

and

$$0 < \epsilon < \int_{e_i} |\gamma_i y(p, t_i)| d|\alpha|.$$

By the preceding theorem $U(\gamma)$ is in E_5 and $U(\gamma_i) \rightarrow U(\gamma)$ in E_5. But since E_6 is a closed linear manifold in E_5 and $U(\gamma_i)$ is in E_6, it follows that $U(\gamma)$ is in E_6 and thus

$$\limsup_i \int_{e_i} |\gamma_i y(p, t_i)|\, d|\alpha| \leqq \lim_i \sup_t \int_E |\gamma y(p, t) - \gamma_i y(p, t)|\, d|\alpha|$$

$$+ \lim_i \sup_t \int_{e_i} |\gamma y(p, t)|\, d|\alpha| = 0,$$

which is a contradiction.

The following two theorems deal with transformations $U(\gamma) = \gamma y_n(t)$ on Γ to E_3 and E_4, respectively. By T_γ will be understood the set in T for which

$$\sum_{n=1}^{\infty} |\gamma y_n(t)| \leqq ||U|| \cdot ||\gamma||.$$

Then by Theorem 1, $T - T_\gamma$ is a null set.

THEOREM 36. *Let $y_n(t)$ be a sequence of functions on T to Y, Γ a separable closed linear manifold in $\overline{Y}$ such that*

(i) *for every set σ of integers there is a $y_\sigma(t)$ on T to Y_0 such that for every γ in Γ,*

$$\sum_{n \epsilon \sigma} \gamma y_n(t) = \gamma y_\sigma(t), \qquad t \epsilon T_\gamma;$$

(ii) *for every γ in Γ, $U(\gamma) = \gamma y_n(t)$ is in E_3; and*

(iii) *for every n and σ the sets $y_n(T)$, $y_\sigma(T)$ are conditionally compact.*

Then it follows that

(iv) *in case $\{\gamma_i\}$ is a bounded sequence in Γ with $\gamma_i y_0 \rightarrow \mu y_0$ on Y_0, $U(\mu)$ is in E_3 and $U(\gamma_i) \rightarrow U(\mu)$ in E_3;*

(v) *U is completely continuous; and*

(vi) *in case Γ is a determining manifold in $\overline{Y}$, there is a null set T_0 such that for every σ we have*

$$\sum_{n \epsilon \sigma} y_n(t) = y_\sigma(t), \qquad t \epsilon T - T_0.$$

THEOREM 37. *If in addition to the conditions of the preceding theorem it is assumed that $U(\gamma)$ is in E_4, then*

$$\lim_N \text{ ess. sup.}_t \sum_N^{\infty} |\gamma y_n(t)| = 0$$

uniformly for $||\gamma|| \leqq 1$. For every σ

$$\sum_{n \epsilon \sigma} y_n(t) = y_\sigma(t)$$

uniformly on $T - T_0$.

To prove Theorems 36 and 37 let $T_0 = \sum_{i=1}^{\infty} (T - T_{\gamma_i})$, where $\{\gamma_i\}$ is dense in Γ so that T_0 is a null set. Now let t be fixed in $T-T_0$, and assume $\epsilon>0$ and N an integer. There is then an i such that

$$\sum_{n=1}^{N} \left| \gamma y_n(t) - \gamma_i y_n(t) \right| < \epsilon;$$

hence

$$\sum_{n=1}^{N} \left| \gamma y_n(t) \right| \leqq \epsilon + \sum_{n=1}^{\infty} \left| \gamma_i y_n(t) \right| \leqq \epsilon + \|U\| \cdot \|\gamma_i\|,$$

which shows that

$$\sup_{t \epsilon T-T_0} \sum_{n=1}^{\infty} \left| \gamma y_n(t) \right| \leqq \|U\| \cdot \|\gamma\|, \qquad \gamma \epsilon \Gamma.$$

Theorems 36 and 37 now follow from Theorems 32 and 33 applied to the set $T-T_0$.

The following two theorems deal with a transformation $U(\gamma)=\gamma y(p, t)$ on Γ to E_7 and E_8, respectively. Here T_γ is the set in T for which

$$\int_E \left| \gamma y(p, t) \right| d\left| \alpha \right|. \leqq \|U\| \cdot \|\gamma\|.$$

By Theorem 1, $T-T_\gamma$ is a null set.

THEOREM 38. *Assume* I *and* II, Γ *separable, and* $U(\gamma)=\gamma y(p, t)$ *on* Γ *to* E_7 *is such that*

(i) *for every* e *in* $\mathfrak{a}(E)$ *there is a* $y_e(t)$ *on* T *to* Y_0 *with* $y_e(T)$ *conditionally compact such that for* γ *in* Γ

$$\gamma y_e(t) = \int_e \gamma y(p, t) d\alpha \quad \text{on} \quad T_\gamma.$$

Then it follows that

(ii) *if* $\{\gamma_i\}$ *is a bounded sequence in* Γ *and* $\gamma_i y \to \mu y$ *for every* y *in* Y_0, *then* $U(\mu)$ *is in* E_7 *and* $U(\gamma_i) \to U(\mu)$ *in* E_7;

(iii) *for almost all* t

$$\lim_{|\alpha|(e)\to 0} \int_e \left| \gamma y(p, t) \right| d\left| \alpha \right| = 0$$

uniformly for $\|\gamma\| \leqq 1$;

(iv) U *is completely continuous*; *and*

(v) *in case* $E=(0, 1)$ *and* α *is Lebesgue measure, then for almost all* t

$$\lim_{h=0} \int_0^{1-h} \left| \gamma y(p+h, t) - \gamma y(p, t) \right| dp = 0$$

uniformly for $\|\gamma\| \leqq 1$.

Let γ_i be dense in Γ and $T_0=\sum T-T_{\gamma_i}$. Fix γ in Γ and t_0 in $T-T_0$, and let $\gamma_i \rightarrow \gamma$. Then

$$\gamma_i y(p, t_0) \rightarrow \gamma y(p, t_0)$$

for every p, and

$$\gamma_i y_e(t) = \int_e \gamma_i y(p, t) d\alpha,$$

providing t is in T_{γ_i}; thus, since t_0 is in T_{γ_i},

$$\gamma_i y_e(t_0) = \int_e \gamma_i y(p, t_0) d\alpha.$$

Since $\gamma_i \rightarrow \gamma$, the integrals $\int_e \gamma_i y(p, t_0) d\alpha$ are equi-absolutely continuous (see [31]) and

$$\int_E | \gamma_i y(p, t_0) - \gamma y(p, t_0) | d | \alpha | \rightarrow 0.$$

Thus

$$\int_E | \gamma y(p, t_0) | d | \alpha | = \lim_i \int_E | \gamma_i y(p, t_0) | d | \alpha | \leqq \|U\| \cdot \|\gamma_i\| \rightarrow \|U\| \cdot \|\gamma\|.$$

This shows that

$$\sup_{t \epsilon T-T_0} \int_E | \gamma y(p, t) | d | \alpha | \leqq \|U\| \cdot \|\gamma\|;$$

and the theorem thus follows from Theorem 34 applied to the set $T-T_0$.

In like manner the following theorem follows from Theorem 35:

THEOREM 39. *If in addition to the assumptions of the preceding theorem we assume that for each* γ *in* Γ

$$\lim_{|\alpha|(e)=0} \text{ess. sup.}_t \int_e | \gamma y(p, t) | d | \alpha | = 0,$$

then this limit exists uniformly for $\|\gamma\| \leqq 1$.

THEOREM 40. *Let* S *be any bounded set in* $\overline{Y}$ *and* $y(e)$ *an additive function on* $\mathfrak{a}(E)$ *to* Y *such that* $y(\mathfrak{a}(E))$ *is separable. Then if for each* γ *in* S

$$\lim_{|\alpha|(e)=0} \gamma y(e) = 0,$$

this same limit holds uniformly for γ *in* S.

For if not, there exist sequences γ_i and e_i with $|\alpha|(e_i)\rightarrow 0$, and a positive ϵ with

$$|\gamma_i y(e_i)| > \epsilon, \qquad i = 1, 2, \cdots.$$

Since $y(\mathfrak{a}(E))$ is separable, there is a subsequence γ_i' of γ_i such that $\gamma_i' y(e)$ converges for each e; thus by a theorem of Saks [31] the functions $\gamma_i' y(e)$ are equi-absolutely continuous, which contradicts the above inequality.

THEOREM 41. *Suppose that the space* $\mathfrak{a}(E)$ *when metrized by the distance function*

$$(e_1, e_2) = |\alpha|(e_1 + e_2 - e_1 e_2)$$

is a separable space. Let Γ *be a determining manifold in* $\overline{Y}$, *and let* Γ^* *be a set of points in* $\overline{Y}$ *(not necessarily in* Γ*) such that for every* γ *in* Γ *there is a sequence* γ_n^* *of finite linear combinations of elements in* Γ^* *with*

$$\lim_n \gamma_n^* y = \gamma y, \qquad y \in Y.$$

Let $y(e)$ *be an additive function on* $\mathfrak{a}(E)$ *to* Y *such that*

(i) *for every sequence* $\{e_n\}$ *of disjoint sets in* $\mathfrak{a}(E)$, $y(\sum e_n)$ *is in the closed linear manifold determined by* $y(e_n)$;

(ii) $\lim_{|\alpha|(e)=0} \gamma^* y(e) = 0$, $(\gamma^* \in \Gamma^*)$.

Then

$$\lim_{|\alpha|(e)=0} y(e) = 0.$$

Since for arbitrary γ in Γ

$$\gamma y(e) = \lim_n \gamma_n^* y(e),$$

and each $\gamma_n^* y(e)$ is continuous on the complete metric space $\mathfrak{a}(E)$, the function $\gamma y(e)$ is in Baire's first class and thus continuous at a point. Since it is additive, it must be continuous everywhere.

Let e_n be a sequence of disjoint sets in $\mathfrak{a}(E)$. Then (i) implies that for every set σ of integers $y(\sum_{n\epsilon\sigma} e_n)$ is in the closed linear manifold determined by $y(e_n)$, $(n \in \sigma)$. Further

$$\sum_{n\epsilon\sigma} \gamma y(e_n) = \gamma y\left(\sum_{n\epsilon\sigma} e_n\right), \qquad \gamma \in \Gamma.$$

Thus it follows from Theorem 32 that $y(e)$ is completely additive. From this fact it follows immediately that $\lim_n y(e_n) = y(e)$ if $e_n \rightarrow e$ monotonically. Now let $\{e_p\}$ be a sequence dense in $\mathfrak{a}(E)$, and let Y_0 be the closed linear manifold in Y determined by $y(e)$, where e is a sum of a finite number of the sets e_p. Then Y_0 is separable. Now let e be an arbitrary set in $\mathfrak{a}(E)$, and from a subsequence of $\{e_p\}$ converging in the metric of $\mathfrak{a}(E)$ to e, pick a subsequence e_n' such that

$$\left(e, \prod_{m=1}^{\infty} \sum_{n=m}^{\infty} e_n'\right) = 0,$$

that is, such that

$$e = \prod_{m=1}^{\infty} \sum_{n=m}^{\infty} e_n'$$

except for a null set. Now $y(\sum_{n=m}^{m+p} e_n')$ is in Y_0 and

$$\lim_{p} y\left(\sum_{n=m}^{m+p} e_n'\right) = y\left(\sum_{n=m}^{\infty} e_n'\right),$$

so that for each m, $y(\sum_{n=m}^{\infty} e_n')$ is in Y_0. But

$$\lim_{m} \sum_{n=m}^{\infty} e_n' = \prod_{m=1}^{\infty} \sum_{n=m}^{\infty} e_n'$$

monotonically, and thus

$$y(e) = \lim_{m} y\left(\sum_{n=m}^{\infty} e_n'\right) \text{ is in } Y_0.$$

Hence by the preceding theorem

$$\lim_{|\alpha|(e)=0} y(e) = \lim_{|\alpha|(e)=0} \sup_{\|\gamma\|=1} \gamma y(e) = 0.$$

THEOREM 42. *Suppose that the space* $\mathfrak{a}(E)$, *when metrized by the distance function*

$$(e_1, e_2) = |\alpha|(e_1 + e_2 - e_1e_2),$$

is a separable space. Let $y(e)$ *be an additive function on* $\mathfrak{a}(E)$ *to* Y, *and let* Γ^* *be a set in* $\overline{Y}$ *such that for every* γ *in* $\overline{Y}$ *there is a sequence* γ_n^* *of finite linear combinations of elements in* Γ^* *with*

$$\lim_{n} \gamma_n^* y = \gamma y, \qquad y \in Y.$$

If

$$\lim_{|\alpha|(e)=0} \gamma^* y(e) = 0, \qquad \gamma^* \in \Gamma^*,$$

then

$$\lim_{|\alpha|(e)=0} y(e) = 0.$$

This follows from the preceding theorem applied to $\Gamma=\overline{Y}$ noting that in this case the equation

$$\sum_n \gamma y(e_n) = \gamma y\left(\sum_n e_n\right), \qquad \gamma\in\overline{Y},$$

where $\{e_n\}$ is a sequence of disjoint sets, assures us that hypothesis (i) of the preceding theorem is fulfilled.

CHAPTER III

3.0. **Extension of linear functionals and integration.** In this section the domain of a point ν in $\overline{X}$ will be extended from X to $\mathfrak{X}$; that is, we shall assign a meaning to νf, where ν is a linear functional on X and f is a point in $\mathfrak{X}=\mathfrak{X}[Y, \Gamma]$ and in particular investigate the properties of the integral $\int_e f d\alpha$.

THEOREM 43. *If f is in $\mathfrak{X}[Y, \Gamma]$, ν is in $\overline{X}$, and γ is in Γ, then $\nu\gamma f(.)$ is linear in γ and*

$$|\nu\gamma f(.)| \leqq \|\nu\|\cdot\|\gamma\|\cdot\|f\|.$$

This is a corollary of Theorem 1. The equation

$$\bar{\gamma}(\gamma) = \nu\gamma f(.)$$

then defines uniquely a point $\bar{\gamma}$ in $\overline{\Gamma}$, and we shall define νf as this point in $\overline{\Gamma}$. The notation is perhaps faulty in that it does not show the dependence of νf upon the closed linear manifold Γ, and perhaps $\nu_\Gamma f$ would be preferable. This latter notation will be used in any discussion where the manifold Γ is not fixed. It might be noted here that if f is in $\mathfrak{X}[Y, \Gamma']$, and $\Gamma'\supset\Gamma$, then $\nu_{\Gamma'} f$ is an extension of $\nu_\Gamma f$.

Besides the space $\mathfrak{X}[Y, \Gamma]$ it will sometimes be useful to consider the subclass $\mathfrak{X}_0[Y, \Gamma]$ consisting of all f in $\mathfrak{X}[Y, \Gamma]$ having the property that for every ν in $\overline{X}$ there is a y (depending on ν and f) such that

$$\nu\gamma f = \gamma y, \qquad \gamma\in\Gamma.$$

THEOREM 44. *The space $\mathfrak{X}_0[Y, \Gamma]$ is linear, and in case $\Gamma=\overline{Y}$ it is closed in $\mathfrak{X}[Y, \Gamma]$.*

To see that $\mathfrak{X}_0$ is closed in $\mathfrak{X}$ if $\Gamma=\overline{Y}$, suppose that f_n is in $\mathfrak{X}_0$, f is in $\mathfrak{X}$, and $\|f_n-f\|\text{I}\to 0$, where the norm is taken in the space $\mathfrak{X}$. This means that

$$\sup_{\|\gamma\|\leqq 1}\|\gamma f_n - \gamma f\| \to 0;$$

hence

$$\lim_n \sup_{\|\gamma\|\leqq 1}\sup_{\|\nu\|\leqq 1}|\nu\gamma f_n - \nu\gamma f| = 0,$$

or

$$\sup_{\|\gamma\|\leqq 1}\ \sup_{\|\nu\|\leqq 1} \left| \gamma y_n(\nu) - \nu\gamma f \right| = \sup_{\|\nu\|\leqq 1} \left\| y_n(\nu) - \bar{\gamma}(\nu) \right\|_{\bar{\Gamma}} \to 0,$$

where $y_n(\nu)$ is regarded as a point in $\bar{\Gamma} = \bar{\bar{Y}}$. Thus since Y is closed in $\bar{\bar{Y}}$ there is a $y(\nu)$ such that $y_n(\nu) \to y(\nu)$ uniformly for $\|\nu\| \leqq 1$ and thus

$$\bar{\gamma}(\nu)\gamma = \nu\gamma f = \gamma y(\nu)$$

for every γ in Γ. The proof that $\mathfrak{X}_0$ is linear will be left to the reader.

THEOREM 45. *If $\Gamma = \bar{Y}$, then a necessary and sufficient condition for a point f in $\mathfrak{X}[Y, \Gamma]$ to be in $\mathfrak{X}_0[Y, \Gamma]$ is that for every ν_0 in a fundamental set $F \subset \bar{X}$ there is a y_0 such that*

$$\nu_0\gamma f(.) = \gamma(y_0), \qquad \gamma \in \Gamma.$$

For if $\nu\gamma f(.) = \gamma(y)$ for every ν in a fundamental set, the same identity holds for every ν in a dense set. Thus if ν is an arbitrary point in $\bar{X}$ and $\|\nu_n - \nu\| \to 0$, where

$$\nu_n\gamma f(.) = \gamma y_n, \qquad \gamma \in \Gamma,$$

we have, using Theorem 43,

$$\|y_m - y_n\| = \sup_{\|\gamma\|\leqq 1} \left| (\nu_n - \nu_m)\gamma f(.) \right| \leqq \|\nu_n - \nu_m\| \cdot \|f\| \to 0,$$

so that if $y = \lim_n y_n$, $\nu\gamma f(.) = \gamma y$.

This proves the sufficiency of the condition, and the necessity is obvious.

3.1. **Integration of numerical functions.** Before proceeding to a discussion of integration of an abstract valued function of an abstract variable it is necessary to set down here certain properties of real summable functions of an abstract variable as well as properties of the Lebesgue integral of such functions.

It is well known from the works of Radon [28], Fréchet [8], Nikodym [22], Ridder [29], and others, that a theory of Lebesgue integration can be developed for real functions of an abstract variable. In fact there are numerous equivalent ways of defining the Lebesgue integral of such a function and several of these are discussed in the paper by Ridder. A basis for such a theory is usually a completely additive family $\mathfrak{a}(E)$ of "measurable" subsets of a given measurable set E and a completely additive real function α. What we have to say here will be based upon the postulates and results in the treatise of Saks ([30], pp. 247–263). Since a completely additive set function is expressible as the difference of two completely additive non-negative set functions, one might restrict the discussion (as Saks does) to the case where α is monotone. We prefer not to assume this. Since only functions which are *summable* with respect to $|\alpha|$, the total variation of α, are considered, there can be no meaningless symbols such as $\infty - \infty$ arising in the discussion.

The symbol L^q, $L^q(E)$, or $L^q(E, \alpha)$ will be used for the space of numerical measurable functions $\phi(p)$ on E for which $|\phi(p)|^q$ is summable with respect to $|\alpha|$ on E; and L, $L(E)$, or $L(E, \alpha)$ will be used in place of L^1, $L^1(E)$, $L^1(E, \alpha)$, respectively. Upon introducing the norm

$$\|\phi\| = \left(\int_E |\phi(p)|^q d|\alpha|\right)^{1/q},$$

the space L^q becomes a Banach space.* The triangle property of this norm follows from Minkowski's inequality for denumerable sums together with the fact that a function in L^q can be approached in norm by denumerably valued functions. The completeness of the space can be established in a well known fashion. It suffices to show that a Cauchy sequence ϕ_n in L^q determines a function ϕ such that $\phi_n(p) \to \phi(p)$ approximately with respect to α on E, and secondly that the integrals $\int_e |\phi_n(p)|^q d|\alpha|$ are equi-absolutely continuous. From these two results one readily concludes that ϕ is in L^q and $\|\phi_n - \phi\| \to 0$.

The general linear functional γ on L^q, $(q < \infty)$, is expressible in terms of a point ψ in $L^{q'}$ (where $1/q + 1/q' = 1$) by the formula

$$\gamma\phi = \int_E \psi(p)\phi(p)d|\alpha|,$$

and the norm $\|\gamma\|$ is given by

$$\|\gamma\| = \left(\int_E |\psi(p)|^{q'} d|\alpha|\right)^{1/q'}, \qquad \text{if } q' < \infty,$$

$$= \text{ess. sup.} \ |\psi(p)|, \qquad \text{if } q' = \infty.$$

To see this we note that by a theorem of Nikodym [22]

$$\gamma\phi_e = \int_e \psi(p)d|\alpha|,$$

where ϕ_e is the characteristic function of the measurable set e. Therefore $\gamma\phi = \int_E \psi(p)\phi(p)d|\alpha|$ for all finitely valued functions ϕ. If ϕ_n is a sequence of finitely valued functions with $\|\phi_n - \phi\| \to 0$, then $\psi(p)\phi_n(p) \to \psi(p)\phi(p)$ approximately with respect to α on E, and $\int_e \psi(p)\phi_n(p)d|\alpha|$ converges for every measurable set e. This shows that $\psi\phi$ is summable on E and that $\gamma\phi = \int_E \psi(p)\phi(p)d|\alpha|$. Furthermore, as the reader can readily show, every

* We assume that $1 \leq q < \infty$. For $q = \infty$ the space L^q is the space $M(E)$ of essentially bounded and measurable functions.

function in L^q is approachable in norm by finitely valued functions, and thus $\gamma\phi = \int_E \psi(p)\phi(p)d|\alpha|$ for every ϕ in L^q.

To see that ψ is in $L^{q'}$ we proceed as follows. Since $\psi\phi$ is in L for every ϕ in L^q, we have $|\psi(p)|^{1/q}\operatorname{sgn}\psi(p)$ in L^q and thus

$$\int_E |\psi(p)|^{1+1/q}d|\alpha| = \gamma(|\psi(p)|^{1/q}\operatorname{sgn}\psi(p)) \leqq \|\gamma\|\left(\int_E |\psi(p)|\,d|\alpha|\right)^{1/q}$$
$$= \|\gamma\|(\gamma(\operatorname{sgn}\psi))^{1/q} \leqq \|\gamma\|^{1+1/q}[|\alpha|(E)]^{1/q},$$

which shows that $|\psi(p)|^{1/q+1/q^2}$ is in L^q. Thus

$$\int_E |\psi(p)|^{1+1/q+1/q^2}d|\alpha| = \gamma(|\psi(p)|^{1/q+1/q^2}\operatorname{sgn}\psi(p))$$
$$\leqq \|\gamma\|\left(\int_E |\psi(p)|^{1+1/q}d|\alpha|\right)^{1/q}$$
$$\leqq \|\gamma\|\cdot\|\gamma\|^{1/q+1/q^2}[|\alpha|(E)]^{1/q^2}$$
$$= \|\gamma\|^{1+1/q+1/q^2}[|\alpha|(E)]^{1/q^2}.$$

In general

$$\int_E |\psi(p)|^{1+1/q+\cdots+1/q^n}d|\alpha| \leqq \|\gamma\|^{1+1/q+\cdots+1/q^n}[|\alpha|(E)]^{1/q^n}.$$

Taking now first the case where $q>1$, we see that the integrand on the left side of the above inequality approaches $|\psi(p)|^{q'}$; hence by Fatou's lemma

$$\int_E |\psi(p)|^{q'}d|\alpha| \leqq \|\gamma\|^{q'}.$$

By Hölder's inequality $\|\gamma\|^{q'} \leqq \int_E |\psi(p)|^{q'}d|\alpha|$, so that the theorem is established for $q>1$. In case $q=1$ let e_m be the set on which $|\psi(p)| \geqq m$; then

$$m^n|\alpha|(e_m) = \int_{e_m} m^n d|\alpha| \leqq \int_E |\psi(p)|^n d|\alpha| \leqq \|\gamma\|^n|\alpha|(E).$$

That is,

$$(m/\|\gamma\|)^n|\alpha|(e_m) \leqq |\alpha|(E),$$

for every positive number m and every integer n. This shows that $|\alpha|(e_m)=0$ if $m>\|\gamma\|$, or in other words,

$$\text{ess. sup. } |\psi(p)| \leqq \|\gamma\|.$$

On the other hand, it is obvious that

$$\text{ess. sup. } |\psi(p)| \geqq \|\gamma\|,$$

which completes the proof of the theorem.

From well known inequalities it follows that for every ψ in $L^{q'}$ and ϕ in L^q the product $\psi\phi$ is in L; and if ψ is fixed, the function $\int_E \psi(p)\phi(p)d|\alpha|$ is a linear functional on L^q. Thus the conjugate space $\bar{L}^q$ is isometrically isomorphic with $L^{q'}$. To summarize the above facts about L^q we state the following theorem:

THEOREM 46. *Assume $1 \leqq q < \infty$. Then the space $L^q = L^q(E) = L^q(E, \alpha)$ of numerical measurable functions ϕ for which $|\phi|^q$ is summable with respect to $|\alpha|$ on E is a Banach space under the norm*

$$\|\phi\| = \left(\int_E |\phi(p)|^q d|\alpha|\right)^{1/q}.$$

Every linear functional γ on L^q is expressible in the form

$$\gamma\phi = \int_E \psi(p)\phi(p)d|\alpha|,^*$$

where ψ is a point of $L^{q'}$, $(q' = q/(q-1))$, and

$$\begin{aligned} \|\gamma\| &= \left(\int_E |\psi(p)|^{q'} d|\alpha|\right)^{1/q'}, && \textit{if } q' < \infty, \\ &= \text{ess. sup. } |\psi(p)|, && \textit{if } q' = \infty. \end{aligned}$$

Conversely if ψ is a point of $L^{q'}$, then $\psi\phi$ is summable for every ϕ in L^q, and $\int_E \psi(p)\phi(p)d|\alpha|$ is a linear functional on L^q.

In case $q = \infty$ the general linear functional on L^q is given in terms of an integral with respect to an additive set function of bounded variation [7], [15], but we shall not use this in what follows.

3.2. **Integration of abstract functions.** In case $X = L^q(E, \alpha)$ and the functional ν in $\bar{X}$ is taken as $\nu\phi = \int_e \phi d\alpha$, then the meaning assigned (in §3.0) to νf for f in $\mathfrak{L}^q(E)[Y, \Gamma]$ is to be taken as the definition of $\int_e f d\alpha$.

Our chief interest in this chapter will be the linear space $\mathfrak{L}^q(E)[Y, \Gamma]$, its specializations $\mathfrak{L}^q(E)[Y, \bar{Y}]$, $\mathfrak{L}^q(E)_0[Y, \bar{Y}]$, and $\mathfrak{L}^q(E)[\bar{Y}, Y] = \mathfrak{L}^q(E)_0[\bar{Y}, Y]$, and the integral as a linear operator on these spaces. Here and elsewhere unless explicitly stated to the contrary q is an arbitrary real number with $1 \leqq q \leqq \infty$. The space $\mathfrak{L}(E)_0[Y, \bar{Y}] = \mathfrak{L}^1(E)_0[Y, \bar{Y}]$ includes the various classes of functions called summable or integrable by the authors Graves [10], Hildebrandt [14], Bochner [4], Dunford [6], Birkhoff [3], and, in view of Theorem 45, is identical with the class of summable functions discussed by

* It is also expressible in the form $\gamma\phi = \int_E \xi(p)\phi(p)d\alpha$.

Pettis [24]. The space $\mathfrak{L}(E)[\bar{Y}, Y]$ is the space of summable functionals defined recently by Gelfand [9] for the case $E=(0, 1)$.

We shall begin with a discussion of the general space $\mathfrak{L}^q(E)[Y, \Gamma]$ and the integral on this class. The discussion can then be applied to all of the above cases. A number of the properties of the space $\mathfrak{L}^q(E)[Y, \Gamma]$, and of the integral $\int_E f(p)d\alpha$ on this space, are immediate consequences of the definitions and of Theorem 1. We shall state a few of them first.

THEOREM 47. *The space* $\mathfrak{L}^q(E)[Y, \Gamma]$ *is a normed linear space, the norm being that of Theorem* 1. *The integral* $\int_E f(p)d\alpha$ *is a linear (that is, additive and continuous) operation on* $\mathfrak{L}^q(E)[Y, \Gamma]$ *to* $\bar{\Gamma}$.

THEOREM 48. (i) *If f is in* $\mathfrak{L}^q(E)[Y, \Gamma]$, *then f is in* $\mathfrak{L}^q(e)[Y, \Gamma]$ *for every e in* $\mathfrak{a}(E)$, *the family of measurable subsets of E.*

(ii) *The function f on E to Y is in* $\mathfrak{L}^q(E)[Y, \Gamma]$ *if and only if* $\phi\cdot f$ *is in* $\mathfrak{L}(E)[Y, \Gamma]$ *for every ϕ in* $L^{q'}(E)$.

THEOREM 49. *If γ is in Γ, ϕ in $L^{q'}(E)$, and f in* $\mathfrak{L}^q(E)[Y, \Gamma]$, *then*

$$\left|\int_E \gamma\phi(p)f(p)d\alpha\right| \leqq \|f\|\cdot\|\phi\|\cdot\|\gamma\|.$$

A corollary of this is the following theorem:

THEOREM 50. *If* $1<q\leqq\infty$, *the integral*

$$\int_e \phi(p)f(p)d\alpha$$

is a completely additive and absolutely continuous function on $\mathfrak{a}(E)$ *for every f in* $\mathfrak{L}^q(E)[Y, \Gamma]$ *and ϕ in* $L^{q'}(E)$.

Here and elsewhere in the paper a set function $y(e)$ is called *absolutely continuous* in case $y(e)\rightarrow 0$ with $|\alpha|(e)$, and *completely additive* in case $y(\sum_n e_n)=\sum_n y(e_n)$ for every sequence $\{e_n\}$ of disjoint measurable sets. In the preceding equality the series on the right must be unconditionally convergent since the left side is independent of the order of the sequence $\{e_n\}$.

THEOREM 51. *If Y is separable and* $q>1$, *then*

$$\mathfrak{L}^q(E)[Y, \bar{Y}] = \mathfrak{L}^q(E)_0[Y, \bar{Y}].$$

For if $\gamma_n y\rightarrow\gamma y$ for every y in Y, then $\|\gamma_n\|$ is a bounded sequence, and, by the preceding theorem, we have for every e in $\mathfrak{a}(E)$

$$\lim_{|\alpha|(e')=0}\int_{ee'}\gamma_n f(p)d\alpha = 0$$

uniformly in n. Thus

$$\int_e \gamma_n f(p)d\alpha \to \int_e \gamma f(p)d\alpha.$$

By a theorem of Banach ([1], p. 131, Theorem 8) there is a y such that

$$\gamma y = \int_e \gamma f(p)d\alpha, \qquad \gamma \in \overline{Y}.$$

The desired conclusion follows from Theorem 45.

Theorem 51 contains a recent theorem of Krein [17]. Krein has proved that if $f(t)$ on $(0, 1)$ to a Banach space Y has the property that $\gamma f(t)$ is continuous for every γ in $\overline{Y}$, then there is a point y in Y such that

$$\gamma y = \int_0^1 \gamma f(t)d\alpha, \qquad \gamma \in \overline{Y},$$

and has used this result in a discussion of a fixed point theorem. In order to see more clearly the connection between Krein's result and Theorem 51, we should state this theorem in a slightly different form. If we do not want all of $\mathfrak{L}^q(E)[Y, \overline{Y}]$ to be in $\mathfrak{L}^q(E)_0[Y, \overline{Y}]$ but merely want a particular function f in $\mathfrak{L}^q(E)[Y, \overline{Y}]$ to be in $\mathfrak{L}^q(E)_0[Y, \overline{Y}]$, it is sufficient to assume that $f(E-$a null set) is separable (and is thus not necessary to assume the whole of Y to be separable), which is equivalent [24] to assuming that f is measurable. Theorem 51 stated in this new form reads as follows:*

THEOREM 52. *Assume $q>1$, and let f on E to Y be measurable and such that γf is in $L^q(E)$ for every γ in $\overline{Y}$. Then f is in $\mathfrak{L}^q(E)_0[Y, \overline{Y}]$, which means that for every ϕ in $L^{q'}(E)$ there is a y_ϕ such that*

$$\gamma y_\phi = \int_E \phi(p)\gamma f(p)d\alpha, \qquad \gamma \in \overline{Y}.$$

To obtain Krein's result from this it suffices to prove that $f[(0, 1)]$ is separable. Let π_n be a partitioning of $(0, 1)$ into intervals δ_n^m, $(m=1, 2, \cdots, p_n)$, such that the norm of the partitioning approaches zero with $1/n$. If τ_n^m is a point of δ_n^m, and $f_n(t)=f(\tau_n^m)$ on δ_n^m, then $\gamma f_n(t) \to \gamma f(t)$ for every γ and t, and thus $f(t)$ must be in the separable closed linear manifold determined by the points $f(\tau_n^m)$.

In the case considered by Krein more might be said, for the boundedness of $\gamma f(t)$ for every γ in $\overline{Y}$ insures us that the norms $\|f(t)\|$ are bounded, and thus the function f is an absolutely integrable and measurable function.

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* Pettis [24] has given a slightly different form of this theorem.

THEOREM 53. *If f is in $\mathfrak{L}(E)[Y, \Gamma]$, and the set of values taken by the integral*

$$\int_e f(p)d\alpha, \qquad e \in \mathfrak{a}(E),$$

is a separable set in $\overline{\Gamma}$, then the integral is a completely additive and absolutely continuous function on $\mathfrak{a}(E)$.

Since the unit sphere in Γ is a bounded set in $\overline{\overline{\Gamma}}$, this theorem is a corollary of Theorem 40.

THEOREM 54. *Suppose $\mathfrak{a}(E)$ separable,* f in $\mathfrak{L}^q(E)_0[Y, \Gamma]$, where Γ is a determining manifold in $\overline{Y}$, and*

(i) *for any sequence $\{e_n\}$ of disjoint sets, $\int_{\Sigma e_n} f(p)d\alpha$ is in the closed linear manifold determined by $\int_{e_n} f(p)d\alpha$.*

Then the integral $\int_e f(p)d\alpha$ is a completely additive and absolutely continuous function on $\mathfrak{a}(E)$. Further if $\Gamma = \overline{Y}$, the hypothesis (i) *may be omitted.*

This is a corollary of Theorems 41 and 42. It is not as general as these theorems for there are absolutely continuous set functions which are not indefinite integrals [24].

THEOREM 55. *Let E be the real interval (a, b) and α Lebesgue measure, and let f be in $\mathfrak{L}(E)[Y, \Gamma]$ and ϕ in $L(E)$. Then if*

$$\bar{\gamma}(t) = \int^t f(s)ds, \qquad \Phi(t) = \int^t \phi(s)ds,$$

we have

$$\phi\bar{\gamma} \in \mathfrak{L}(E)[\overline{\Gamma}, \Gamma], \qquad \Phi f \in \mathfrak{L}(E)[Y, \Gamma],$$

and

$$\int_a^b \phi(t)\bar{\gamma}(t)dt = \Phi\bar{\gamma}\Big|_a^b - \int_a^b \Phi(t)f(t)dt.$$

Let $\mathfrak{a}(E')$ be another σ-field of measurable subsets of a measurable set E', and let α' be a completely additive set function on $\mathfrak{a}(E')$.

THEOREM 56. *If the function $f(p, p')$ on EE' to Y belongs to the space† $\mathfrak{L}(EE', \alpha\times\alpha')[Y, \Gamma]$, and if for all p in E, except for a set upon which $|\alpha| = 0$, $f(p, p')$ is in $\mathfrak{L}(E', \alpha')[Y, \Gamma]$, then $\int_{E'} f(p, p')d\alpha'$ is in $\mathfrak{L}(E, \alpha)[\overline{\Gamma}, \Gamma]$, and*

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* For the case $\Gamma = \overline{Y}$ Pettis proves this theorem without the assumption of separability. His method however can be applied to the theorem as stated here without the assumption of separability.

† For the notion of the product measure on the product space and the general Fubini theorem for real functions see Saks [30], or Łomnicki and Ulam [20].

$$\int_{EE'} f(p, p')d(\alpha \times \alpha') = \int_E d\alpha \int_{E'} f(p, p')d\alpha'.$$

THEOREM 57. *Let f_n be in $\mathfrak{L}^q(E)[Y, \Gamma]$, $(n=1, 2, \cdots)$, and suppose $\int_e f_n(p)d\alpha$ is absolutely continuous for each n. Then if*

$$\lim_n \int_e f_n(p)d\alpha$$

exists for every e in $\mathfrak{a}(E)$, the integrals $\int_e f_n(p)d\alpha$ are equi-absolutely continuous.

The familiar argument of Saks [31] based on the Baire category theorem holds in this environment. For upon setting $F_n(e) = \int_e f_n(p)d\alpha$, the space $\mathfrak{a}(E)$ with metric $(e_1, e_2) = |\alpha|(e_1 + e_2 - e_1e_2)$ is the sum of the closed sets

$$\mathfrak{a}_q \equiv \mathfrak{a}(E)[\|F_m(e) - F_n(e)\| \leqq \epsilon/3, \ m, n \geqq q],$$

and thus by the Baire theorem, there is an integer q_0 and a sphere $S(e_0, r) \subset \mathfrak{a}_{q_0}$. Let $\delta < r$ be such that

$$\|F_{q_0}(e)\| < \epsilon/3, \qquad |\alpha|(e) < \delta.$$

Thus for $|\alpha|(e) < \delta$ the sets $e_1 = e + (e_0 - e)$, $e_2 = e_0 - e$ are in $S(e_0, r)$, and

$$\|F_m(e) - F_n(e)\| \leqq \|F_m(e_1) - F_n(e_1)\| + \|F_m(e_2) - F_n(e_2)\| \leqq 2\epsilon/3,$$

for $m, n \geqq q_0$; and for $|\alpha|(e) < \delta$ we have

$$\|F_m(e)\| \leqq \epsilon, \qquad m \geqq q_0.$$

THEOREM 58. *Let f_n be in $\mathfrak{L}(E)[Y, \Gamma]$, $(n=1, 2, \cdots)$, let $f_n(p) \to f(p)$ approximately on E, and let $\int_e f_n(p)d\alpha$ be absolutely continuous for each n. Then the following assertions are equivalent:*

(i) *The limit,* $\lim \int_e f_m(p)d\alpha$, *exists on* $\mathfrak{a}(E)$.

(ii) *The function f is in* $\mathfrak{L}(E)[Y, \Gamma]$ *and*

$$\lim_m \int_e f_m(p)d\alpha = \int_e f(p)d\alpha$$

uniformly on $\mathfrak{a}(E)$.

(iii)
$$\lim_{|\alpha|(e)=0} \limsup_m \left| \int_e f_m(p)d\alpha \right| = 0.$$

(iv)
$$\lim_{|\alpha|(e)=0} \int_e f_m(p)d\alpha = 0$$

uniformly in m.

The proof of the theorem is not a great deal different than for the case of absolutely integrable and measurable functions $f(p)$ (see Dunford [6], pp. 447–448). There are several differences however; for example, in the present case we do not know that the set $e(n, \epsilon)$ consisting of all the points of e where $\|f_n(p)-f(p)\| \geqq \epsilon$ is a measurable set. It might also be pointed out that in the case of absolutely integrable functions it is necessary to assume that the limit function is summable, while in the broader space $\mathfrak{L}(E)[Y, \Gamma]$ this is part of the conclusion. Before proving the theorem we shall recall the meaning of approximate convergence. The sequence $f_n(p)$ is said to approach $f(p)$ approximately on E in case for every n and $\epsilon>0$ there is a measurable set $e'(n, \epsilon)>e(n, \epsilon)$ such that

$$\lim_n |\alpha| (e'(n, \epsilon)) = 0.$$

If $\|\gamma\|=1$, and $e(\gamma, n, \epsilon)$ is that part of e on which

$$|\gamma f_n(p) - \gamma f(p)| \geqq \epsilon,$$

then $e(\gamma, n, \epsilon)$ is measurable and

$$e(\gamma, n, \epsilon) \subset e(n, \epsilon) \subset e'(n, \epsilon),$$

which shows that $\gamma f_n(p) \rightarrow \gamma f(p)$ in measure. The sequence $f_n(p)$ is said to approach $f(p)$ almost uniformly on E if for every $\epsilon>0$ there are sets E_ϵ and E_ϵ' with E_ϵ' measurable, $E_\epsilon' \supset E-E_\epsilon$, $|\alpha|(E_\epsilon')<\epsilon$, and $f_n(p)\rightarrow f(p)$ uniformly on E_ϵ. It is known [6] that if $f_n(p)\rightarrow f(p)$ approximately on E, then every subsequence of $f_n(p)$ contains a subsequence which approaches $f(p)$ almost uniformly on E. Now to demonstrate the theorem we shall show the following implications: (iii)→(i)→(iv)→(ii)→(iii). Assuming (iii), we have for every $\epsilon>0$ a $\delta>0$ such that for every measurable e with $|\alpha|(e)<\delta$ there is an n_e such that

$$\left\| \int_e f_n(p) d\alpha \right\| < \epsilon, \qquad n \geqq n_e.$$

Let it first be supposed that $f_n(p)$ converges almost uniformly on E. Then

$$\left\| \int_E (f_m(p) - f_n(p)) d\alpha \right\| \leqq \left\| \int_{E-E'(\delta)} (f_m(p) - f_n(p)) d\alpha \right\| + \left\| \int_{E'(\delta)} (f_m(p) - f_n(p)) d\alpha \right\|,$$

where $E'(\delta) \supset E-E(\delta)$, $|\alpha|(E'(\delta))<\delta$, and $f_n(p)$ converges uniformly on $E(\delta)$ and hence on $E-E'(\delta)$. There is an m_1 such that

$$\left\| \int_{E-E'(\delta)} (f_m(p) - f_n(p)) d\alpha \right\| < \epsilon,$$

$$\left\| \int_{E'(\delta)} (f_m(p) - f_n(p)) d\alpha \right\| < 2\epsilon, \qquad m, n \geqq m_1.$$

Thus for m, $n \geqq m_1$ we have

$$\left\| \int_E (f_m(p) - f_n(p)) d\alpha \right\| < 3\epsilon.$$

The same proof holds for an arbitrary e in $\mathfrak{a}(E)$. The same conclusion follows if f_n only approaches f approximately, since then every subsequence of $\int_e f_n(p)d\alpha$ contains a subsequence approaching $\int_e f(p)d\alpha$. Thus (iii)→(i). The implication (i)→(iv) follows from the preceding theorem. To show that (iv)→(ii), first note that since $\lim_{|\alpha|(e)=0} \int_e \gamma f_m(p) d\alpha = 0$ uniformly in m, and $\gamma f_m \to \gamma f$ approximately on E, the function γf must be in $L(E)$; hence f is in $\mathfrak{L}(E)[Y, \Gamma]$. Furthermore for each γ we have

$$\text{(a)} \qquad \lim_m \int_e \gamma f_m(p) d\alpha = \int_e \gamma f(p) d\alpha.$$

To complete the proof of (ii) we have to show that (a) holds uniformly for $\|\gamma\| \leqq 1$ and e in $\mathfrak{a}(E)$. Now

$$\left\| \int_e (f_m(p) - f_n(p)) d\alpha \right\| \leqq \left\| \int_{e-E'(m,n,\epsilon)} (f_m(p) - f_n(p)) d\alpha \right\| + \left\| \int_{eE'(m,n,\epsilon)} (f_m(p) - f_n(p)) d\alpha \right\|,$$

where $E'(m, n, \epsilon)$ is a measurable set covering the set $E(m, n, \epsilon)$ upon which $\|f_m(p) - f_n(p)\| > \epsilon$. Also

$$\left\| \int_{e-E'(m,n,\epsilon)} (f_m(p) - f_n(p)) d\alpha \right\| = \sup_{\|\gamma\| \leqq 1} \left| \int_{e-E'(m,n,\epsilon)} \gamma [f_m(p) - f_n(p)] d\alpha \right| \leqq \epsilon |\alpha| (E).$$

There is a $\delta > 0$ such that if $|\alpha|(e) < \delta$,

$$\left\| \int_e (f_m(p) - f_n(p)) d\alpha \right\| < \epsilon, \qquad m, n = 1, 2, \cdots,$$

and an m_0 such that

$$|\alpha|\,(E'(m, n, \epsilon)) < \delta, \qquad n, m \geqq m_0.$$

Consequently, since

$$|\alpha|\,(eE'(m, n, \epsilon)) \leqq |\alpha|\,(E'(m, n, \epsilon)) < \delta, \qquad n, m \geqq m_0,$$

we have

$$\left\| \int_e f_m(p)d\alpha - \int_e f_n(p)d\alpha \right\| \leqq \epsilon[1 + |\alpha|\,(E)], \qquad n, m \geqq m_0;$$

and, in view of (a), this proves (ii). From (ii) it follows that $\int_e f(p)d\alpha$ is absolutely continuous. Thus

$$\lim_{|\alpha|(e)=0} \limsup_{m} \left\| \int_e f_n(p)d\alpha \right\| = \lim_{|\alpha|(e)=0} \left\| \int_e f(p)d\alpha \right\| = 0;$$

hence (ii) implies (iii), which completes the proof of the theorem.

THEOREM 59. *If f is in $\mathfrak{L}^q(E)[Y, \overline{Y}]$, and if for every e in $\mathfrak{a}(E)$ there is a y in Y such that*

$$\gamma y = \int_e \gamma f(p)d\alpha, \qquad \gamma \epsilon \overline{Y},$$

then for every ϕ in $L^{q'}(E)$ there is a y in Y such that

$$\gamma y = \int_E \gamma\phi(p)f(p)d\alpha.$$

This is a corollary of Theorem 45 in the case where $1 \leqq q < \infty$. The case $q = \infty$ is the case where $\gamma f(p)$ is measurable and $|\gamma f(p)|$ is essentially bounded for every γ in $\overline{Y}$. The integral

$$\int_E \phi(p)f(p)d\alpha,$$

which we know always exists in $\overline{\overline{Y}}$, must be in Y, for if ϕ_n is a sequence of finitely valued functions (that is, functions assuming only a finite number of values) with

$$\|\phi_n - \phi\| = \int_E |\phi_n(p) - \phi(p)|\, d\,|\alpha| \to 0,$$

then $\int_E \phi_n(p)f(p)d\alpha$ is in Y, and by Theorem 49,

$$\left\| \int_E \phi_n(p)f(p)d\alpha - \int_E \phi(p)f(p)d\alpha \right\| \leqq \|\phi_n - \phi\| \cdot \|f\| \to 0,$$

so that $\int_E \phi(p)f(p)d\alpha$ is also in Y.

THEOREM 60. *For f in $\mathfrak{L}(E)_0[Y, \overline{Y}]$ ($\mathfrak{L}(E)[\overline{Y}, Y]$), the operation $U(\gamma) = \gamma f(.)$ on $\overline{Y}$ $[Y]$ to $L(E)$ is weakly continuous. If f is in $\mathfrak{L}(E)_0[Y, \overline{Y}]$, and $f(E - a$ null set) is a separable set in Y (or if f is measurable*), then U is completely continuous.*

For if $\gamma_i y \to \gamma y$ for every y, the integrals

$$\int_e \gamma_i f(p) d\alpha = \gamma_i y_e$$

are equi-absolutely continuous, and since $\gamma_i f(p) \to \gamma f(p)$ for each p, we have

$$\|U(\gamma_i) - U(\gamma)\| = \int_E |\gamma_i f(p) - \gamma f(p)| \, d|\alpha| \to 0.$$

A similar proof holds for the case where f is in $\mathfrak{L}(E)[\overline{Y}, Y]$. The complete continuity of U is a corollary of Theorem 34.

THEOREM 61. *If f is in $\mathfrak{L}^q(E)_0[Y, \overline{Y}]$ and ϕ is in $L^q(E)$, then a necessary and sufficient condition that there exist a γ in $\overline{Y}$ with*

(i) $$\|\gamma\| \leqq M, \qquad \gamma f(p) = \phi(p),$$

almost everywhere on E, is that

(ii) $$\left| \int_E \eta(p)\phi(p) d\alpha \right| \leqq M \left\| \int_E \eta(p) f(p) d\alpha \right\|$$

for every η in a set dense in $L^{q'}(E)$.

If there does exist a γ satisfying (i), then (ii) is obviously true. To prove the converse† first note that if (ii) holds for every η in a set dense in $L^{q'}$, then by Theorem 49 it holds for every η in $L^{q'}$. Let Y_0 be the linear manifold in Y determined by

$$\int_E \eta(p) f(p) d\alpha, \qquad \eta \in L^{q'}.$$

Now if

$$y = \int_E \eta(p) f(p) d\alpha = \int_E \eta'(p) f(p) d\alpha,$$

we have

* That is, the limit almost everywhere of a sequence of finitely valued functions. In view of a result of Pettis [24] and the fact that γf is measurable, the assumption that $f(E - $a null set) is separable is equivalent to the assumption that f is measurable.

† The proof parallels Banach's discussion of the moment problem ([1], p. 55, Theorem 4).

$$\left|\int_E \eta(p)\phi(p)d\alpha - \int_E \eta'(p)\phi(p)d\alpha\right| \leqq M \left\|\int_E [\eta(p) - \eta'(p)]f(p)d\alpha\right\| = 0,$$

so that

$$\gamma_0(y) = \int_E \eta(p)\phi(p)d\alpha$$

is a well defined additive functional on Y_0. It is also continuous since

$$|\gamma_0 y| = \left|\int_E \eta(p)\phi(p)d\alpha\right| \leqq M \left\|\int_E \eta(p)f(p)d\alpha\right\| = M\|y\|.$$

Hence by the Hahn-Banach theorem on the extension of linear functionals there is a linear functional γ defined on the whole of Y with

$$|\gamma y| \leqq M\|y\|, \qquad y \in Y,$$

$$\gamma y = \gamma_0 y = \int_E \eta(p)\phi(p)d\alpha, \qquad y \in Y_0.$$

Thus for every η in $L^{q'}(E)$

$$\int_E \eta(p)\phi(p)d\alpha = \gamma \int_E \eta(p)f(p)d\alpha = \int_E \eta(p)\gamma f(p)d\alpha.$$

Upon writing $\psi(p) = \phi(p) - \gamma f(p)$, we have

$$\int_E \eta(p)\psi(p)d\alpha = 0, \qquad \eta \in L^{q'}.$$

According to a theorem of Hahn-Sierpiński ([30], p. 249) every E' in $\mathfrak{a}(E)$ is representable as the sum of two disjoint sets E_+', E_-' such that α is non-negative on $\mathfrak{a}(E_+')$ and non-positive on $\mathfrak{a}(E_-')$. It follows immediately that $\psi(p) = 0$ almost everywhere.

THEOREM 62. *Let f_n be in $\mathfrak{L}(E)_0[Y, \overline{Y}]$, $(n = 1, 2, \cdots)$, and $f_n(p) \to f(p)$ approximately on E. Then the following assertions are equivalent*:

(i) *The limit*, $\lim_m \int_e f_m(p)d\alpha$, *exists on* $\mathfrak{a}(E)$.

(ii) *The function f is in* $\mathfrak{L}(E)_0[Y, \overline{Y}]$ *and*

$$\lim_m \int_e f_m(p)d\alpha = \int_e f(p)d\alpha$$

uniformly on $\mathfrak{a}(E)$.

(iii) $$\lim_{|\alpha|(e)=0} \limsup_m \left\|\int_e f_m(p)d\alpha\right\| = 0.$$

(iv) *Uniformly with respect to n,*

$$\lim_{|\alpha|(e)=0} \int_e f_n(p)d\alpha = 0.$$

In view of Theorems 54 and 58 there is only one thing to prove here; namely, that f is in $\mathfrak{L}(E)_0[Y, \overline{Y}]$. This however follows immediately from Theorem 59 together with the facts that Y is closed in $\overline{\overline{Y}}$ and

$$\lim_n \int_e f_n(p)d\alpha = \int_e f(p)d\alpha.$$

THEOREM 63. *Let E be the real interval (a, b), α Lebesgue measure, ϕ real and continuous on (a, b), and f in $\mathfrak{L}(E)_0[Y, \overline{Y}]$ or $\mathfrak{L}(E)[\overline{Y}, Y]$. Then if*

$$\beta(u) = \int_a^u f(v)dv,$$

the Riemann-Stieltjes integral $\int_a^b \phi(u)d\beta(u)$ exists and

$$\int_a^b \phi(u)d\beta(u) = \int_a^b \phi(u)f(u)du.$$

That $\int_a^b \phi(u)d\beta(u)$ exists in the Riemann-Stieltjes sense follows from Theorems 11, 12, and 13; and the equality follows from the corresponding Theorem for numerical functions.

THEOREM 64. *Let E, α be as in the preceding theorem, and let f be in $\mathfrak{L}(E)_0[Y, \overline{Y}]$. If Y is separable, then for every real function Φ absolutely continuous on $a \leqq t \leqq b$ we have almost everywhere,*

$$\frac{d}{ds}\left[\Phi(s)\int_a^s f(t)dt - \int_a^s f(t)\Phi(t)dt\right] = \Phi'(s)\int_a^s f(t)dt.$$

This is not an immediate corollary of Theorem 55 as one might think at first sight. By that theorem F is in $\mathfrak{L}(E)[\overline{\overline{Y}}, \overline{Y}]$, where

$$F(p) = \Phi'(p)\int_a^p f(t)dt$$

and

$$\int_a^s F(p)dp = \Phi(s)\int_a^s f(t)dt - \int_a^s \Phi(s)f(s)ds.$$

Since f is in $\mathfrak{L}(E)_0[Y, \overline{Y}]$, it follows that F is in $\mathfrak{L}(E)[Y, \overline{Y}]$. This is still not enough however, for an indefinite integral of a function even in $\mathfrak{L}(E)_0[Y, \overline{Y}]$ is not necessarily differentiable. It will be true that

$$\frac{d}{ds}\int_a^s F(p)dp = F(s)$$

almost everywhere, (which is what we want to show) if $F(p)$ is measurable and $\|F(p)\|$ is summable, that is, if F belongs to the class of absolutely integrable and measurable functions (see Bochner [4]). By Theorem 2, $\int_a^p f(t)dt$ is bounded on (a, b), and for each γ in $\overline{Y}$ the function $\gamma\int_a^p f(t)dt$ is measurable (in fact absolutely continuous) on (a, b). Since Y is separable, it follows from a theorem of Pettis [24] that $\int_a^p f(t)dt$ is measurable. Thus $F(p)$ is the product of a real summable function and a bounded measurable function and hence is measurable and absolutely integrable.

THEOREM 65. *Let E, α, f, Y be as in the preceding theorem.* If f is defined to be constant outside of (a, b) and $f_h(t) = f(t+h)$, then* $\lim_{h=0} f_h - f = 0$, *that is,*

$$\lim_{h=0}\int_a^b |\gamma f(t+h) - \gamma f(t)|\, dt = 0$$

uniformly for $\|\gamma\| \leqq 1$.

This follows immediately from Theorem 34 by taking the class T as a class consisting of a single element.

CHAPTER IV

4.0. **Instances.** It is not the purpose of the present chapter to apply the preceding results, as we intend to do that later, but merely to call attention to some special instances of a few of the theorems.

As an instance of Theorem 10 we mention the following theorem which is a generalization of a theorem of Hahn-Steinhaus [12], given by Saks and Tamarkin [32].

THEOREM 66. *Let V be an arbitrary set (v) and U a set (u) in which there is a notion of null set satisfying condition* (N). *If for every u in U (or almost all u in U) and every v in V the function $H(t, u, v)$ is of bounded variation on $a \leqq t \leqq b$ and normalized so that†*

$$H(t, u, v) = 1/2[H(t+0, u, v) + H(t-0, u, v)],$$

* In both of these theorems it is not entirely necessary that Y be separable. All that is needed is that $f(E-$a null set) be separable which, in view of Pettis' result, is equivalent to saying that f is measurable.

† Or normalized in any other manner so that its total variation is the same as the norm of the linear functional it defines on C.

then if, for every continuous function ϕ defined on (a, b), there is a null set $U_\phi \subset U$ and an M_ϕ such that

$$\int_a^b \phi(t)d_tH(t, u, v) < M_\phi, \qquad u \in U - U_\phi, v \in V,$$

there will be a null set $U_0 \subset U$ and a constant M such that

$$\int_a^b | d_tH(t, u, v) | < M, \qquad u \in U - U_0, v \in V.$$

This corresponds to taking $Y = BV$, $\Gamma = C$. Another choice might be $Y = M$, $\Gamma = L$. This would yield the theorem:

THEOREM 67. *Let U and V be as in the preceding theorem, and $H(t, u, v)$ essentially bounded in t for almost all u and all v. Then if for every summable function ϕ there is a null set $U_\phi \subset U$ and a constant M_ϕ such that*

$$\int_a^b \phi(t)H(t, u, v)dt < M, \qquad u \in U - U_\phi, v \in V,$$

there is a null set $U_0 \subset U$ and a constant M such that

$$\operatorname{ess.\,sup.}_t | H(t, u, v) | < M, \qquad u \in U - U_0, v \in V.$$

THEOREM 68. *Let $\mathfrak{a}(E)$, $\mathfrak{a}(E')$ be two families consisting of all measurable subsets of the measurable sets E and E', respectively. Let α, α' be positive, finite, completely additive set functions on $\mathfrak{a}(E)$, $\mathfrak{a}(E')$, respectively. Let $K(p, p')$ be a real function on EE' such that the integral*

$$\int_{e'} K(p, p')d\alpha', \qquad e' \in \mathfrak{a}(E'),$$

exists for almost all (with respect to α) p in E and is essentially bounded and measurable on E. If for every ϕ in $L(E, \alpha)$ there is a constant M_ϕ such that

$$\left| \int_E \phi(p)d\alpha \int_{e'} K(p, p')d\alpha' \right| \leqq M_\phi \alpha'(e'),$$

then for every ψ in $L(E', \alpha')$ the integral

$$\int_{E'} K(p, p')\psi(p')d\alpha'$$

exists for almost all p, is essentially bounded and measurable in p, and

$$\lim_{\psi = 0} \operatorname{ess.\,sup.}_p \left| \int_{E'} K(p, p')\psi(p')d\alpha' \right| = 0.$$

In Theorem 18 take $Z=L(E', \alpha')$, $Y=L^\infty(E, \alpha)$, $\Gamma=L(E, \alpha)$, and

$$f(z) = \int_{E'} K(p, p')z(p')d\alpha'.$$

By assumption $f(z)$ is defined for all finitely valued functions, so that $D(f)$ is dense in $L(E', \alpha')$. Also by assumption

$$|\gamma f(z)| \leqq M_\gamma \|z\|$$

providing z is the characteristic function of a measurable set e'. It is readily shown that this same inequality holds for any finitely valued z, and the adjoint of f is thus defined for every γ in Γ. Then by Theorem 18, $f(z)$ can be extended to be linear and bounded on $L(E', \alpha')$. Let z_n be finitely valued functions approaching z in $L(E', \alpha')$. Then there exists a set $E_0 \epsilon\, \mathfrak{a}(E)$ with $\alpha(E-E_0)=0$ such that

$$\int_{E'} K(p, p')z_n(p')d\alpha'$$

converges uniformly with respect to p in E_0. Furthermore for each p in E_0

$$K(p, p')z_n(p') \to K(p, p')z(p)$$

approximately with respect to α' on E', and

$$\text{ess. sup.}_p \left| \int_{e'} K(p, p')z_n(p')d\alpha' \right| = \|f(z_n\psi_{e'})\| \leqq \|f\| \int_{e'} |z_n(p')| d\alpha',$$

where $\psi_{e'}$ is the characteristic function of e'. This shows that the integrals $\int_{e'}K(p, p')z_n(p')d\alpha'$ are, at least for p in a set of measure $\alpha(E)$, equi-absolutely continuous, and thus $\int_E K(p, p')z(p')d\alpha'$ exists almost everywhere and is essentially bounded. The last conclusion follows from Theorem 18.

THEOREM 69. *Let Z be a Banach space and $f(z)=K(z, t)$ an additive function on Z to $L_q(0, 1)$. If*

$$\lim_{z=0} \int_0^s K(z, t)dt = 0, \qquad 0 \leqq s \leqq 1,$$

then

$$\lim_{z=0} \text{ess. sup.}_t |K(z, t)| = 0, \qquad \text{if } q = \infty,$$

$$\lim_{z=0} \left(\int_0^1 |K(z, t)|^q dt \right)^{1/q} = 0, \qquad \text{if } 1 \leqq q < \infty.$$

In case $1<q\leqq\infty$, this is Theorem 21 applied to $L^{q'}(0, 1)$. For $q=1$ one

can use Theorem 20 with $Y=L$, $\Gamma=C$ = continuous functions, Γ^* = characteristic functions of intervals.

THEOREM 70. *Let Z be a Banach space, and $f(z)=K(z, t)$ an additive function on Z to the space $\overline{C}$ (where C is the space of continuous functions on* (0, 1)). *If*

$$\lim_{z=0} K(z, t) = 0, \qquad 0 \leqq t \leqq 1,$$

then

$$\lim_{z=0} \int_0^1 | d_t K(z, t) | = 0.$$

This is Theorem 20 applied to the case where $Y=\overline{C}$, $\Gamma=C$, Γ^* = characteristic functions of intervals.

Assume $p>1$; then an instance of Theorem 34 (or 60), if we use the fact that if $Y=\overline{\overline{Y}}$, then $\mathfrak{L}(E)[Y, \overline{Y}]=\mathfrak{L}(E)_0[Y, \overline{Y}]$, is the following theorem:

THEOREM 71. *If for almost all s in* (0, 1) *the function $H(s, t)$ is in $L^{p'}(0, 1)$, and if*

$$U(\phi) = \psi(s) = \int_0^1 H(s, t)\phi(t)dt$$

is in $L(0, 1)$ for every ϕ in $L^p(0, 1)$, then the operation $U(\phi)$ is completely continuous on L^p to L.

Likewise we have the following theorem:

THEOREM 72. *If for almost all s in* (0, 1) *the sequence $a_n(s)$ is in $l^{p'}$ and*

$$U(x) = \sum_{n=1}^{\infty} \xi_n a_n(s) = \psi(s)$$

is in L (0, 1) *for every vector $x=\{\xi_n\}$ in l^p, then the operation $U(x)$ on l^p to L is completely continuous.*

As instances of Theorem 32 we have, if T is a class with one element and we take first $Y=l^{p'}$, $\Gamma=l^p$ and then $Y=L^{p'}$, $\Gamma=L^p$, the following two theorems:

THEOREM 73. *Every linear operator from l^p, $(p>1)$, to l is completely continuous.*

THEOREM 74. *Let $a_n(s)$ be in $L^{p'}(0, 1)$, $(n=1, 2, \cdots)$, and suppose that*

$$U(\phi) = \int_0^1 a_n(s)\phi(s)ds$$

is in l for every ϕ in L^p. Then $U(\phi)$ is completely continuous.

As a matter of fact *any continuous linear operator from $\bar{Y}$ to l is completely continuous provided $Y=\bar{\bar{Y}}$.* This fact has been proved by Pettis [25] and follows also from Theorem 30 together with the remarks following Theorem 32.

THEOREM 75. *If $\{y_n\}$ is a sequence of points in a Banach space Y such that $\sum_n \xi_n y_n$ converges for every $x=\{\xi_n\}$ in m, then the operation*

$$U(x) = \sum_{n=1}^{\infty} \xi_n y_n$$

*on m to Y is completely continuous.**

By hypothesis all partial sums of $\sum_n y_n$ are convergent; hence by a result of Orlicz [23] the series $\sum_n y_n$ is unconditionally convergent. The operation $V(\gamma)=\{\gamma y_n\}$ is therefore a linear operation on $\bar{Y}$ to l. By Theorem 32, V is completely continuous, and since U is the adjoint of V, it follows from a result of Schauder [33] that U is completely continuous also.

If we had been working in the complex domain instead of the real domain, the following theorem would be an immediate corollary of the definition of the integral. As it is, a few words of explanation are necessary. In this final theorem Y is a complex Banach space, that is, a complete normed vector space satisfying all the postulates for a Banach space with the real number system replaced by the complex number system (see Wiener [38]). Let $\bar{Y}$ be the space of all complex valued continuous linear functionals on Y. Then upon placing

$$\|\gamma\| = \sup_{\|y\|=1} |\gamma y|$$

for γ in $\bar{Y}$, the space $\bar{Y}$ is also a complex Banach space.

The space Y may be regarded as a Banach space; thus ([1], p. 55) for each y_0 there is a real function μ such that

$$\mu(y_0) = \|y_0\|, \qquad \sup_{\|y\|=1} |\mu y| = 1,$$

$$\mu(c_1 y_1 + c_2 y_2) = c_1\mu(y_1) + c_2\mu(y_2), \qquad c_1, c_2 \text{ real}.$$

The function μ is not in $\bar{Y}$ however because $\mu c y \neq c\mu y$ for c complex. However by defining $\mu'(y) = -\mu(iy)$, the function

$$\gamma y = [1/(2)^{1/2}][\mu(y) + i\mu'(y)]$$

is continuous and linear, and $\gamma c y = c\gamma y$ for complex values of c. Thus γ is in $\bar{Y}$. It is readily shown that

$$\|\gamma\| \leqq 1, \qquad |\gamma y_0| \geqq \|y_0\|/(2)^{1/2},$$

* This theorem was proved for the case $Y=l$ by Littlewood [19].

so that

$$\sup_{\|\gamma\|=1} |\gamma y| \geqq \|y\|/(2)^{1/2}, \qquad y \in Y.$$

A *determining manifold* $\Gamma \subset \overline{Y}$ is defined to be a closed linear manifold in $\overline{Y}$ such that

$$\sup_{\|\gamma\|=1} |\gamma y| \geqq M\|y\|, \qquad y \in Y,$$

where γ is restricted to be in Γ and M is some positive constant independent of y.

It is readily shown that a function $y(t)$ on an arbitrary range T is bounded if $\gamma y(t)$ is bounded for every γ in a determining manifold $\Gamma \subset \overline{Y}$. For then Γ is the sum of the closed sets

$$\Gamma_n = \Gamma[\,|\gamma y(t)| \leqq m, t \in T],$$

and by the Baire category theorem there is a sphere $S(\gamma_0, r)$ of Γ contained in Γ_{n_0} for some integer n_0. Thus if $\|\gamma\| < r$,

$$|\gamma y(t)| \leqq |(\gamma - \gamma_0)y(t)| + |\gamma_0 y(t)| \leqq 2n_0, \qquad \text{on} \quad T,$$

so that, for any γ in Γ with $\gamma \ \leqq 1$, $|\gamma y(t)| \leqq 2n_0 r$ on T and thus

$$\|y(t)\| \leqq 2n_0 r/M, \qquad \text{on} \quad T.$$

In terms of this notation we have the following theorem:

THEOREM 76. *Suppose D is a simply connected open set in the complex plane, Y is an arbitrary complex Banach space, and $f(z)$ on D to Y is such that $\gamma f(z)$ is analytic for every γ in Γ. Then $\gamma f(z)$ is analytic for every γ in $\overline{Y}$ and**

(i) *$\int_C f(z)dz = 0$ for any rectifiable curve C in D;*

(ii) *$f(z) = (1/2\pi i)\int_C [f(\zeta)/(\zeta - z)]d\zeta$ if C contains z;*

(iii) *$f(z)$ has strong derivatives of all orders and*

(iv) *$f^{(n)}(z) = (n!/2\pi i)\int_C [f(\zeta)/(\zeta - z)^{n+1}]d\zeta$ if C contains z;*

(v) *the Taylor expansion*

$$\sum_{n=0}^{\infty} \frac{(z-\zeta)^n}{n!} f^{(n)}(z)$$

converges uniformly to $f(z)$ for z in any circle $|z - \zeta| \leqq r$ inside D.

All of the above integrals can be taken in the Riemann sense.

To prove (ii) note first that the boundedness of $\gamma f(z)$ on C for every γ in Γ implies the boundedness of $\|f(z)\|$ on C. Now from the formula

* If the boundary of D is a rectifiable Jordan curve, and $\gamma f(z)$ is continuous in the closed domain $\overline{D}$ for each γ in Γ, then the curve C, appearing in the various integrals, may be taken as the bounding curve.

$$\gamma[f(z+h)-f(z)] = \frac{1}{2\pi i}\int_C \gamma f(\zeta)\left[\frac{1}{\zeta-(z+h)}-\frac{1}{\zeta-z}\right]d\zeta$$

and the boundedness of $\|f(z)\|$ on C it follows that

$$\lim_{h=0}\gamma[f(z+h)-f(z)] = 0$$

uniformly for $\|\gamma\| \leqq 1$, that is, $\lim_{h=0} f(z+h) = f(z)$. Thus the integral

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

exists in the Riemann sense. Since $\gamma f(z) = \gamma g(z)$ for every γ in a determining manifold, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

The remaining conclusions of the theorem follow in the usual manner from the above formula.

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